# LEVEL II

79-13T

①

⑥ MAP ERROR BIT DECODING OF
CONVOLUTIONAL CODES

by ⑨ Doctoral thesis,

⑩ James Thomas/Kindle

⑫ 186p.

⑭ AFIT-CI-79-13T

A Dissertation Presented to the

FACULTY OF THE GRADUATE SCHOOL

UNIVERSITY OF SOUTHERN CALIFORNIA

In Partial Fulfillment of the

Requirements for the Degree

DOCTOR OF PHILOSOPHY

(Electrical Engineering)

⑪ August 1977

012 200

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER<br><br>CI 79-13T | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle)<br><br>Map Error Bit Decoding of Convolutional Codes | | 5. TYPE OF REPORT & PERIOD COVERED<br><br>Dissertation |
| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s)<br><br>James Thomas Kindle | | 8. CONTRACT OR GRANT NUMBER(s) |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br><br>AFIT Student at the University of Southern California | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br><br>AFIT/CI<br>WPAFB OH 45433 | | 12. REPORT DATE<br><br>August 1977 |
| | | 13. NUMBER OF PAGES<br><br>176 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) | | 15. SECURITY CLASS. (of this report)<br><br>UNCLASSIFIED |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release, Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

APPROVED FOR PUBLIC RELEASE AFR 190-17.

JOSEPH P. HIPPS, Major, USAF
Director of Information, AFIT

NOV 9 1978

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

11 21 030

UNIVERSITY OF SOUTHERN CALIFORNIA
THE GRADUATE SCHOOL
UNIVERSITY PARK
LOS ANGELES, CALIFORNIA 90007

*This dissertation, written by*

James Thomas Kindle

*under the direction of h..is.... Dissertation Committee, and approved by all its members, has been presented to and accepted by The Graduate School, in partial fulfillment of requirements of the degree of*

DOCTOR OF PHILOSOPHY

*Dean*

Date September 2, 1977

DISSERTATION COMMITTEE

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R. A. Johnson

To my wife Joan, and

sons James and Michael

## ACKNOWLEDGEMENT

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## Chapter 1

## INTRODUCTION

Since Shannon [1] showed that it is theoretically possible
through coding to achieve reliable communication at rates below chan-
nel capacity, much work in the field of coding theory was centered on
finding and applying various suitable coding and decoding techniques
for memoryless channels. One such technique is convolutional coding,
first discussed by Elias [2] in 1955 and since shown to be quite cost
effective in terms of error correcting capability provided for a given
equipment complexity level. The space shuttle communications system
[3] and commercial communication satellites [4] are but two of
many current examples of the use of convolutional coding to obtain
performance gain in a practical digital communication link having
restrictions on available power.

The convolutional encoding operation is deterministic and thus
quite easily implemented. The decoding of convolutional codes, on the
other hand, must provide correction of random errors made in trans-
mission and thus is a design problem with an underlying statistical
nature. To approach such a problem one ideally would like to apply
statistical estimation theory techniques directly and thus derive an
optimal estimator (decoder). In practice a feasible decoder such as
that due to Viterbi [5] may be developed first and only subsequently

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shown to be an optimal estimator, in this case the optimal maximum likelihood data sequence estimator (Forney[6]). However Massey [7] effectively looked at a Maximum Aposteriori Probability (MAP) error bit estimator and showed how for the orthogonal subset of convolutional codes this could be specialized to the implementable threshold decoding technique.

In this work an attempt is made to first consider the convolutional decoding problem from an optimal estimation viewpoint and then specialize to possibly suboptimal but implementable decoding techniques. While much work on data estimation has appeared subsequent to the Viterbi algorithm, here error estimation is stressed and thus is more akin to the original work of Massey [7].

Chapter 2 provides a brief review of convolutional coding and decoding for memoryless channels with emphasis on only those elements needed later to discuss error estimation and syndrome decoding. Provided is a qualitative breakout of the various possible decoding approaches for data and errors from a probabilistic viewpoint.

Chapter 3 displays general optimal error bit and error sequence estimation techniques both with and without the practical restriction on time delay allowable before an error estimate must be produced. The complementary relationship between MAP error decoding and MAP data decoding is also noted.

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Chapter 4 then concentrates on one of the optimal decoding approaches from Chapter 3, namely Optimal Finite Delay MAP Error Bit decoding. Two suboptimal approximations, ROMMAP and SUB-MAP, are derived and their implementations discussed.

Chapter 5 illustrates how the actual performance of a ROMMAP decoder may be computed and that of the SUBMAP decoder both upper and lower bounded. Also indicated is the relationship between SUB-MAP decoding and conventional feedback decoding. Lastly, simulation results of practical SUBMAP decoder implementations for several convolutional codes are presented to quantitatively illustrate and confirm previous theoretical predictions.

Chapter 6 qualitatively summarizes the conclusions that may be drawn from this investigation and suggests some continuations that may be worthwhile.

## Chapter 2

## CONVOLUTIONAL CODES AND PROBABILISTIC
## DECODING

In this chapter a brief introduction to convolutional coding and
decoding is provided stressing only those areas necessary to later
derivations and implementation. The basic communication system
problem model is first presented to provide motivation for the use of
convolutional coding. Next, various decoding approaches for both
data and error estimation are qualitatively reviewed from a prob-
abilistic standpoint to show the range of options that could be pursued
for implementation of the crucial decoding step. Chapter 3 will further
consider the error estimation options and derive appropriate optimal
decoding techniques.

### 2.1 Basic Communication System Model

The underlying communication system problem can be simply
stated by referring to Figure 2.1. Here a Data Source generates
digital data to be reliably transferred to a Data User through a real
communication channel which is noisy. The output of the source in
this work is assumed to be a data sequence of symbols from GF(q)
as is the input to the user. The noisy channel causes errors to be
introduced into the data as presented to the user with such a frequency
(error rate) so as to make the received data unacceptable. The noisy

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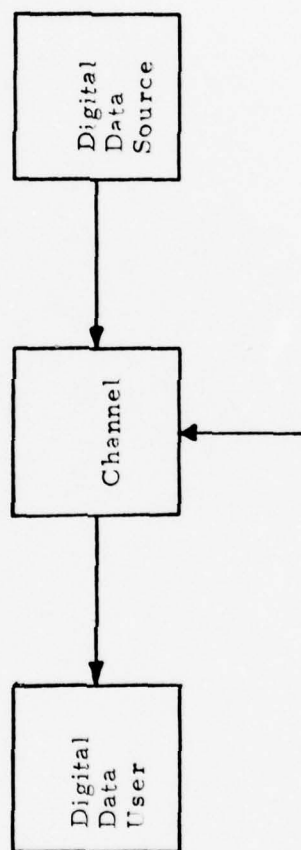
Figure 2.1 Communication System Problem

channel includes all the modulation, transmission, and demodulation of a real communications system.

To reduce the frequency of errors presented to the user, an encoder and decoder are appended to the communication system as shown in Figure 2.2. To simplify analysis, the binary case of GF(q) = GF(2) is assumed and the channel is modeled as an equivalent Binary Symmetric Channel (BSC) as shown in Figure 2.3. This is both a simple and appropriate channel model for a system which employs the commonly used 180 degree Binary Phase Shift Keying (BPSK) modulation with hard decisions in the receiver and which is corrupted by Additive White Gaussian Noise (AWGN). The specific relationship between the BSC crossover probability p and the Signal-to-Noise Ratio (SNR) encountered at the receiver will be given in Section 5.2. The BSC is memoryless in that the symbol output at each discrete time instant is statistically dependent only on the corresponding input symbol and not on any other input or output symbol preceding or following. Thus the BSC is also described as a DMC (Discrete Memoryless Channel).

In this work it also is assumed that the input data rate to the channel is so far below the channel capacity that sufficient bandwidth exists to enable the coding gains predicted in later sections. This is commonly called an infinite bandwidth assumption.

Included in the "solution" of Figure 2.2 are as yet unspecified

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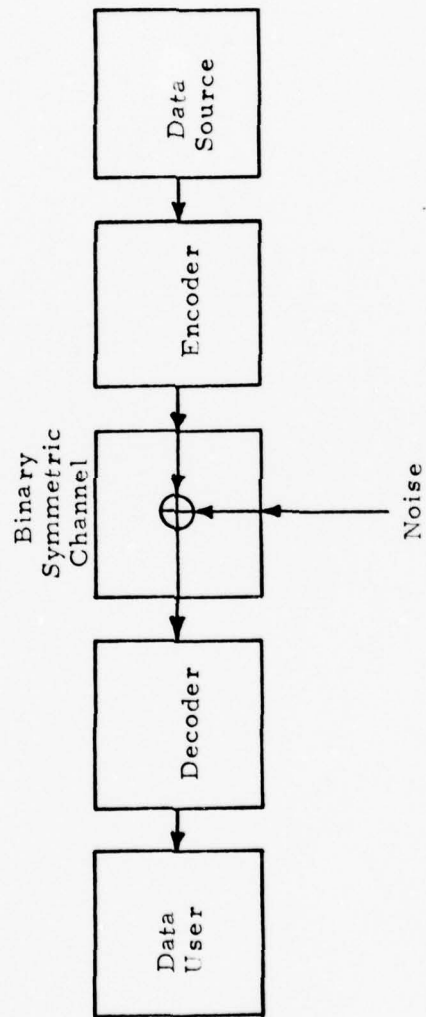
Figure 2.2 Communication System Solution

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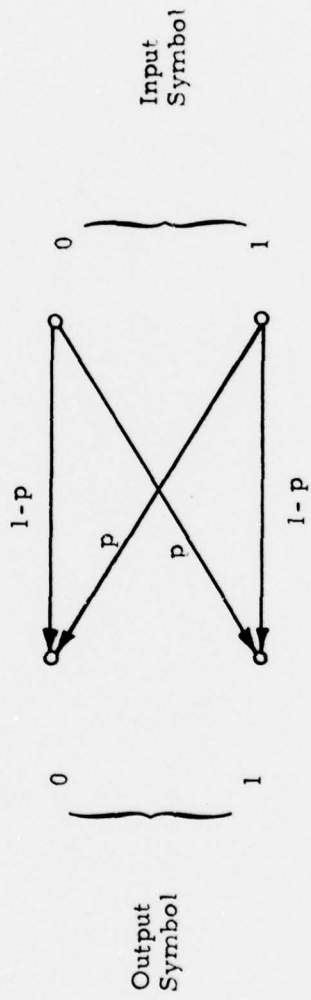
Figure 2.3.  Binary Symmetric Channel (BSC)

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encoder and decoder units. The function of the encoder is to map

data source output symbols into code symbols such that BSC channel

errors made in transmitting such code symbols can sometimes be

corrected by the decoder - thus decreasing the data error rate pre-

sented to the user.

In this work convolutional encoding will be assumed to implement

the encoder unit. This is a relatively simple encoding method. The

more difficult problem of implementing a corresponding decoder will

be investigated using several estimation approaches.

2. 2 <u>Convolutional Encoding</u>

A convolutional encoder is defined as a linear finite-state ma-

chine employing one or more $\nu$ - stage shift registers and having k

inputs at each discrete time instant with n outputs determined by n

linear algebraic function generators operating on the present and

possible $\nu$ prior inputs.

The idea of a simple convolutional encoder is depicted in Figure

2. 4. where each block of k binary bits from the Data Source is en-

coded into an output block of n (n > k) bits to be transferred through

the BSC to the decoder. However, the output n-tuple depends not only

on the current input k-tuple $(I_t^{(1)}, \ldots, I_t^{(k)})$ but also is constrained by

certain prior k-tuples whose effect is still present in the encoder.

Thus the <u>constraint length</u> K of the particular encoder is defined as

the maximum number of n-tuple output bits which can be influenced
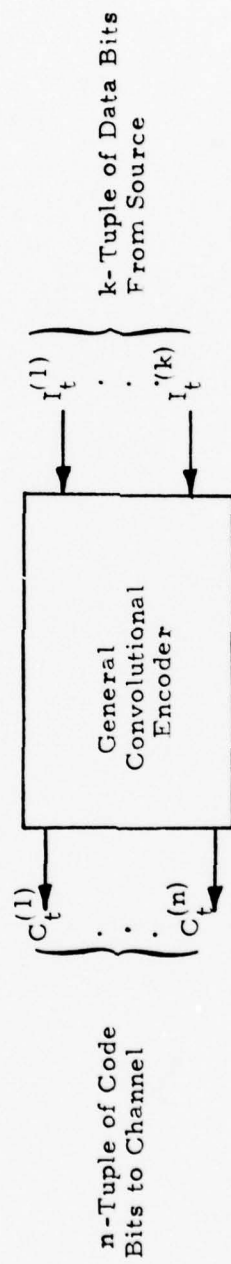
Figure 2.4. Convolutional Encoder

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by an input k-tuple bit. Each bit of the n-tuple is thus generated by a linear combination of K or less data bits. The <u>rate</u> R of the code can be approximated by R = k/n. That is, though the channel is transferring n bits per time period, the <u>information</u> rate is only k bits per time period. The presence of these "excess" n-k bits represents a feature to be exploited later by the decoder to correct BSC transmission errors.

The linear function generator relationship between input k-tuples $(I_t^{(1)}, \ldots, I_t^{(k)})$ and output n-tuples $(c_t^{(1)}, \ldots, c_t^{(n)})$ can in general be expressed in terms of coefficients of <u>generator polynomials</u> in arbitrary variables y and z (Berlekamp [8] ). Using notation similar to that of Berlekamp, the generator polynomial set for k inputs and n outputs

$$G^{(j)}(y, z) = \sum_{\ell=1}^{k} \sum_{h=0}^{\nu} G_{\ell, h}^{(j)} y^{\ell} z^{h} \qquad (2.1)$$

$$\text{for } 1 \leq j \leq n$$

yields the n-tuple output digits (code digits) computed by

$$C_t^{(j)} = \sum_{\ell=1}^{k} \sum_{h=0}^{\nu} G_{\ell, h}^{(j)} I_{t-h}^{\ell} \quad \text{for } 1 \leq j \leq n. \qquad (2.2)$$

Here t is the current time. The variable y is used to index through the input k-tuple positions. The variable z is used to index through information bits within the constraint length of the code. Since the current k-tuple $(I_t^{(1)}, \ldots, I_t^{(k)})$ is available at the encoder input but not

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yet stored, the memory length $\nu$ of the encoder specifies the maximum number of prior input k-tuples which may affect a given $c_t^{(j)}$.

Although the above notation is sufficiently general to be used by all rate k/n convolutional codes, it is worthwhile to specialize to the systematic rate 1/n code case. Thus with k = 1 and D substituted for z, (2.1) and (2.2) become:

$$G^{(j)}(D) = \sum_{h=0}^{\nu} G_h^{(j)} D^h \quad \text{for } 1 \leq j \leq n \tag{2.3}$$

and

$$C_t^{(j)} = \sum_{h=0}^{\nu} G_h^{(j)} I_{t-h} \quad \text{for } 1 \leq j \leq n. \tag{2.4}$$

As an example, consider the binary, rate $R = k/n = \frac{1}{2}$ systematic convolutional encoder shown in Figure 2.5a in terms of binary unit delay (storage) devices and polynomial coefficient weights. The term systematic here means that one code bit, $C_t^{(1)}$, is always identical to $I_t^{(1)}$. The binary logic implementation is shown in Figure 2.5b. In this example $K = \nu + 1 = 5$.

The set of generator polynomials is thus:

$$G^{(1)}(D) = \sum_{h=0}^{\nu} G_h^{(1)} D^h \tag{2.5}$$

$$= G_0^{(1)} + G_1^{(1)} D + G_2^{(1)} D^2 + G_3^{(1)} D^3 + G_4^{(1)} D^4$$

$$= 1 \text{ and}$$

$$G^{(2)} D = \sum_{h=0}^{\nu} G_h^{(2)} D^h$$

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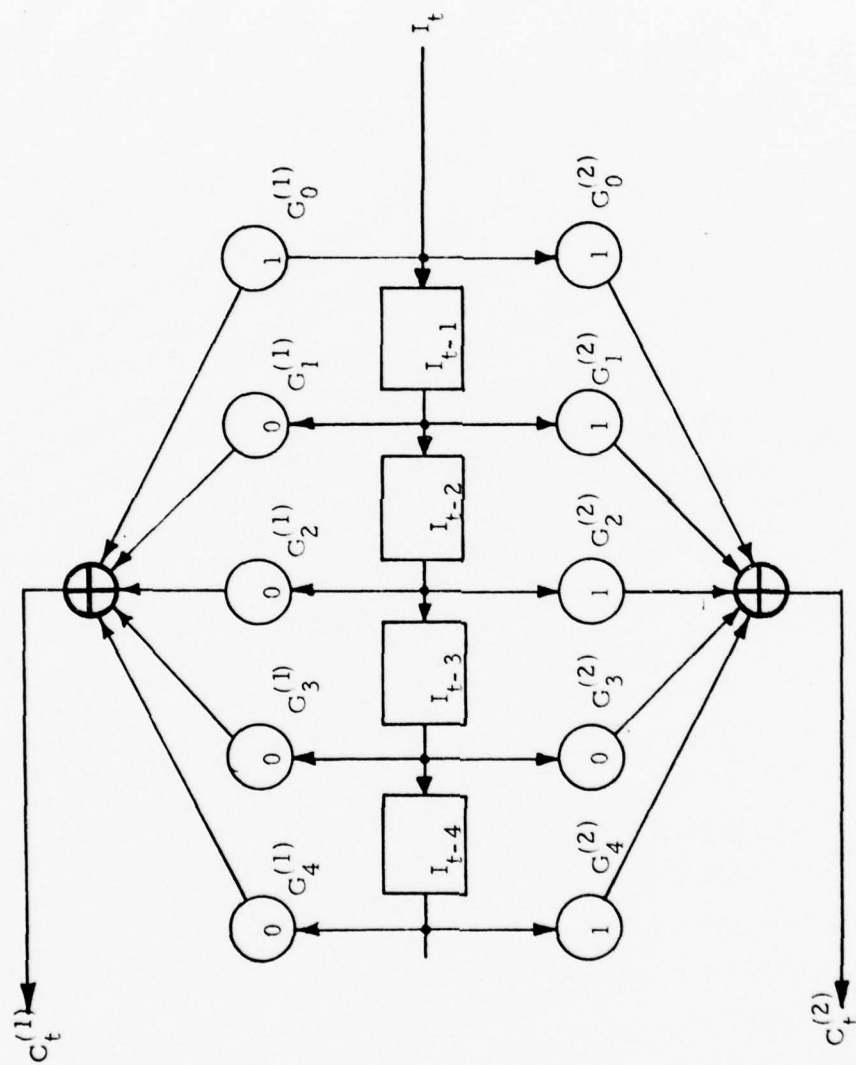
Figure 2.5a  Encoder for K=5 Rate 1/2 Systematic Code

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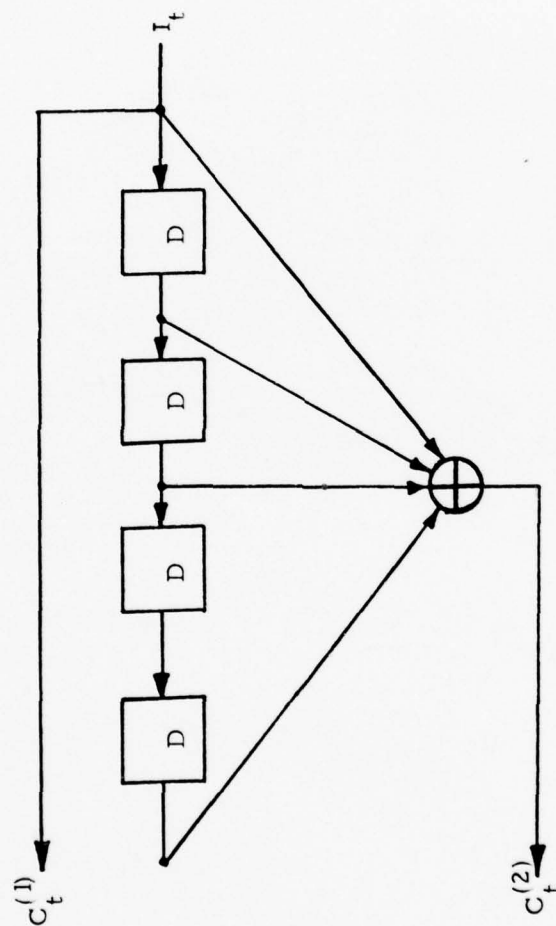
Figure 2.5b  Logic of K=5 Rate 1/2 Systematic Code

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$$= G_0^{(2)} + G_1^{(2)} D + G_2^{(2)} D^2 + G_3^{(2)} D^3 + G_4^{(2)} D^4 \qquad (2.6)$$

$$= 1 + D + D^2 + 0 + D^4 .$$

The output code digits are then:

$$C_t^{(1)} = \sum_{h=0}^{\nu} G_h^{(1)} I_{t-h}$$

$$= G_0^{(1)} I_t + G_1^{(1)} I_{t-1} + G_2^{(1)} I_{t-2} + G_3^{(1)} I_{t-3}$$

$$+ G_4^{(1)} I_{t-4}$$

$$= I_t \qquad (2.7)$$

and

$$C_t^{(2)} = \sum_{h=0}^{\nu} G_h^{(1)} I_{t-h}$$

$$= G_0^{(2)} I_t + G_1^{(2)} I_{t-1} + G_2^{(2)} I_{t-2} + G_3^{(2)} I_{t-3}$$

$$+ G_4^{(2)} I_{t-4} \qquad (2.8)$$

$$= I_t + I_{t-1} + I_{t-2} + 0 + I_{t-4} .$$

In terms of vector notation one would have

$$\underline{C}_t = \begin{bmatrix} C_t^{(1)} \\ C_t^{(2)} \end{bmatrix} \quad \text{and} \quad \underline{I}_t = [I_t] . \qquad (2.9)$$

For this example consider the transmission of the vector $\underline{C}_t$ through the BSC. Each $C_t^{(i)}$ will be independently affected by noise

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in the form of additive binary bit errors $e_t^{(i)}$. Thus at the decoder the received vector will be:

$$\underline{R}_t = \begin{bmatrix} r_t^{(1)} \\ r_t^{(2)} \end{bmatrix} = \underline{C}_t + \underline{E}_t = \begin{bmatrix} C_t^{(1)} + e_t^{(1)} \\ C_t^{(2)} + e_t^{(2)} \end{bmatrix} \tag{2.10}$$

Thus $\underline{E}_t = \underline{R}_t - \underline{C}_t = \underline{R}_t \oplus \underline{C}_t$ in this binary case.

It is instructive at this point to very briefly mention the concepts of encoder state and the encoder state transition probability. The state of the encoder at time t, $\mathscr{A}_t$, is defined for a rate 1/n by

$$\mathscr{A}_t = (I_{t-1}, \ I_{t-2}, \ldots, I_{t-\nu}) \tag{2.11}$$

systematic code. Thus the encoder state can be represented by a binary number composed of the values residing in each of the $\nu$ physical storage elements of the encoder at time t. The encoder state transition probability at time t is

$$P(\mathscr{A}_{t+1} / \mathscr{A}_t),$$

the probability that the next state of the encoder will be $\mathscr{A}_{t+1}$ given that the encoder is now in state $\mathscr{A}_t$.

With such definitions, the operation of the rate 1/n encoder can be considered in terms of a state diagram of $2^\nu$ states. The encoder resides in a current state $\mathscr{A}_t$ and will transition to one of two allowed successor states depending on the value of $I_t$. For illustrative pur-

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poses the simple $\nu = 2, K = 3$ rate $\frac{1}{2}$ systematic encoder of Figure 2.6a is considered from a state diagram point of view in Figure 2.6b.

The output code digits are deterministic functions of both the current state $\mathscr{A}_t$ and the input $I_t$. Since $I_t$ becomes the first element of $\mathscr{A}_{t+1}$, the output digits are equivalently deterministic functions of the state pair $\mathscr{A}_t$, $\mathscr{A}_{t+1}$. Thus a <u>conditional</u> <u>output</u> <u>probability</u> $P(\underline{C}_t / \mathscr{A}_t, I_t) = P(\underline{C}_t / \mathscr{A}_t, \mathscr{A}_{t+1})$ can be defined for any $\underline{C}_t$. Obviously for the <u>encoder</u> this probability is either 0 or 1. However in later discussions it will be shown that for another linear finite-state machine, namely the syndrome former portion of a decoder, the output probability will be more a function of certain random inputs. For now it is only worth noting that the <u>conditional</u> probability $P(\underline{C}_t / \mathscr{A}_t, I_t)$ is independent of all prior encoder states $\mathscr{A}_j$ for $j < t$. Therefore,

$$P(\underline{C}_t / \mathscr{A}_0, \mathscr{A}_1, \ldots, \mathscr{A}_{t-1}, \mathscr{A}_t, \mathscr{A}_{t+1}) = P(\underline{C}_t / \mathscr{A}_t, \mathscr{A}_{t+1}). \qquad (2.12)$$

Thus the encoder can be considered as a Markov source having $2^\nu$ <u>states</u> whose state transitions are determined by the value of the input data $I_t$ through <u>state</u> <u>transition</u> <u>probabilities</u> $P(\mathscr{A}_{t+1} / \mathscr{A}_t)$ and whose outputs are specified by <u>output</u> <u>probabilities</u> $P(\underline{C}_t / \mathscr{A}_t, \mathscr{A}_{t+1})$. This <u>encoder</u> Markov property was used productively by McAdam [9] and Bahl et al. [10] in analyzing encoder operation. It will be applied and expanded in later sections of this investigation to analyze a slightly more complex decoder problem.

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$C_t^{(1)}$

$D$ ← $D$ ← $I_t$

$\oplus$

$C_t^{(2)}$

Figure 2.6a  Simple $\nu = 2$ Encoder

0/00

00

0/01          1/11

1/10

10                01

0/00

0/01          1/11

11

1/10

$\mathcal{S}_t$   Current State at time t

$I_t / C_t^{(1)} C_t^{(2)}$

$\mathcal{S}_{t+1}$

Next State at time t+1
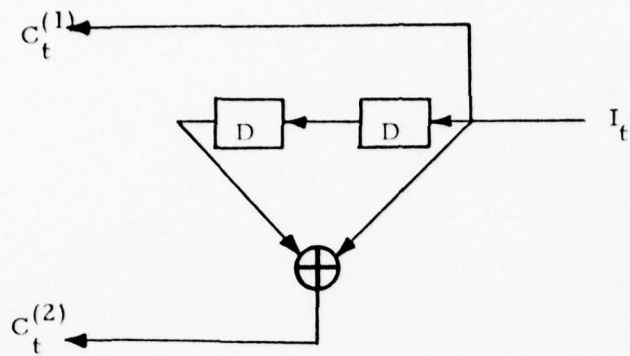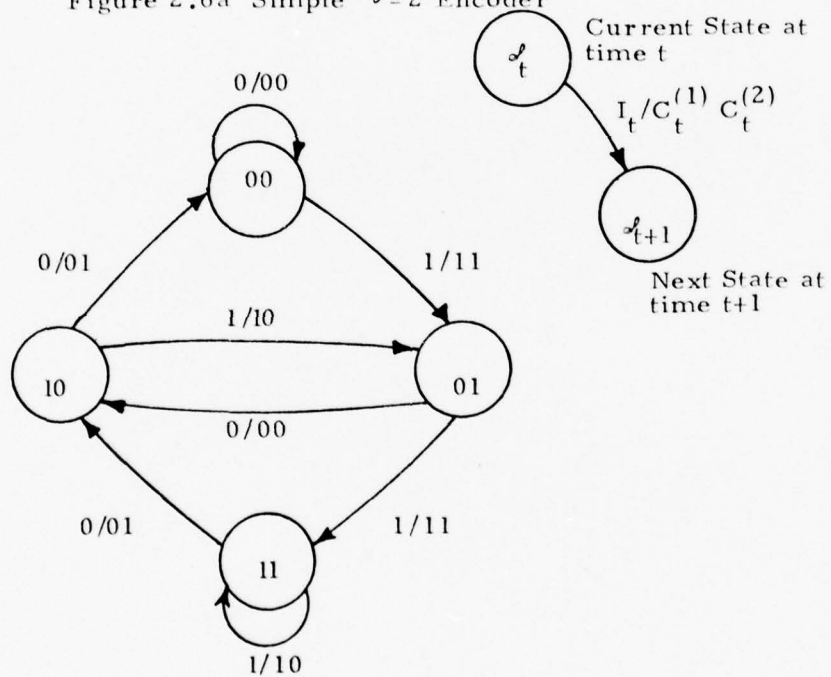
Figure 2.6b  Equivalent State Diagram

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## 2.3 Syndromes

For the moment reconsider the general $k, n$ encoder case. In accordance with Massey [7] (Theorem 9) and Berlekamp [8] one can define syndrome digits $Z_t^{(j)}$ as

$$Z_t^{(j)} = r_t^{(j)} - \sum_{\ell=1}^{k} \sum_{h=0}^{\nu} G_{\ell,h}^{(j)} \, r_{t-h}^{(\ell)} \quad \text{for } k < j \leq n. \quad (2.13)$$

Using Eq. (2.10) this can be expanded to:

$$Z_t^{(j)} = C_t^{(j)} + e_t^{(j)} - \sum_{\ell=1}^{k} \sum_{h=0}^{\nu} G_{\ell,h}^{(j)} \, (C_{t-h}^{(\ell)} + e_{t-h}^{(\ell)}) \quad (2.14)$$

and reduced by Eq. (2.2) to

$$Z_t^{(j)} = e_t^{(j)} - \sum_{\ell=1}^{k} \sum_{h=0}^{\nu} G_{\ell,h}^{(j)} \, e_{t-h}^{(\ell)} \quad \text{for } k < j \leq n. \quad (2.15)$$

In general usage, syndromes are quantities computed from noisy received data but are functions of errors only. Thus it is evident that the syndrome digits above are functions only of the errors introduced by the BSC.

For the binary rate $1/n$ systematic code case the syndrome digits are

$$Z_t^{(j)} = e_t^{(j)} + \sum_{h=0}^{\nu} G_h^{(j)} \, e_{t-h}^{(1)} \quad \text{for } k < j \leq n. \quad (2.16)$$

A vector notation common in later sections for rate $1/n$ codes is

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$$\underline{Z}_t = \begin{bmatrix} Z_t^{(2)} \\ . \\ . \\ . \\ Z_t^{(n)} \end{bmatrix} \tag{2.17}$$

For the previously mentioned $K = 5$ rate $\frac{1}{2}$ example of Eqs. (2.5)

and (2.6) and Figure 2.5 there is only one syndrome digit equation:

$$Z_t^{(j)} = e_t^{(j)} + \sum_{h=0}^{\nu} G_h^{(j)} e_{t-h}^{(1)} \quad \text{(i. e. } j{=}2 \text{ only).} \tag{2.18}$$

Thus,

$$Z_t^{(2)} = e_t^{(2)} + G_0^{(2)} e_t^{(1)} + G_1^{(2)} e_{t-1}^{(1)} + G_2^{(2)} e_{t-2}^{(1)} + G_3^{(2)} e_{t-3}^{(1)} +$$

$$G_4^{(2)} e_{t-4}^{(1)}$$

$$= e_t^{(2)} + e_t^{(1)} + e_{t-1}^{(1)} + e_{t-2}^{(1)} + e_{t-4}^{(1)} \tag{2.19}$$

The actual computation of Eq. (2.18) or Eq. (2.19) is accomplished

by a <u>syndrome former</u> whose physical inputs are received digits $r_t^{(j)}$

$1 \le j \le n$ and whose outputs are syndrome digits $Z_t^{(j)}$ $k < j \le n$. However,

since Eq. (2.15) has shown that the syndrome digits are <u>independent</u>

of the actual code bits transmitted, the analysis of the syndrome

former can be simplified by considering its inputs to be <u>only</u> error

bits $e_t^{(j)}$, $1 \le j \le n$ introduced by the BSC. For example, Figure 2.7a

shows the actual physical syndrome former implementation for the

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K = 5 rate $\frac{1}{2}$ code mentioned previously. Figure 2.7b on the other hand

shows the syndrome former that will be analyzed in lieu of that in

Figure 2.7a. One can also refer to Figure 2.8 which indicates just

where the syndrome former is used in relation to other components

of the convolutional code communication system.

Since Eq. (2.16) applies at a particular time t at the decoder, one

can write a syndrome equation set for any code by considering all t

values in some span, $t_1 \le t \le t_2$. For the example K = 5 code, if one

considers time values including the current time t and 5 prior time

instants, the following syndrome equation set is obtained:

$$Z_{t-5} = e_{t-5}^{(2)} + e_{t-5}^{(1)} + e_{t-6}^{(1)} + e_{t-7}^{(1)} + 0 + e_{t-9}^{(1)}$$

$$Z_{t-4} = e_{t-4}^{(2)} + e_{t-4}^{(1)} + e_{t-5}^{(1)} + e_{t-6}^{(1)} + 0 + e_{t-8}^{(1)}$$

$$Z_{t-3} = e_{t-3}^{(2)} + e_{t-3}^{(1)} + e_{t-4}^{(1)} + e_{t-5}^{(1)} + 0 + e_{t-7}^{(1)}$$

$$Z_{t-2} = e_{t-2}^{(2)} + e_{t-2}^{(1)} + e_{t-3}^{(1)} + e_{t-4}^{(1)} + 0 + e_{t-6}^{(1)}$$

$$Z_{t-1} = e_{t-1}^{(2)} + e_{t-1}^{(1)} + e_{t-2}^{(1)} + e_{t-3}^{(1)} + 0 + e_{t-5}^{(1)}$$

$$Z_t = e_t^{(2)} + e_t^{(1)} + e_{t-1}^{(1)} + e_{t-2}^{(1)} + 0 + e_{t-4}^{(1)} \tag{2.20}$$

Note how a particular error digit (e.g. $e_{t-5}^{(1)}$) influences a span

of syndrome digits (in this case $Z_i$ for $t-1 \le i \le t-5$). This will be

significant in later decoding steps where it will be convenient to define

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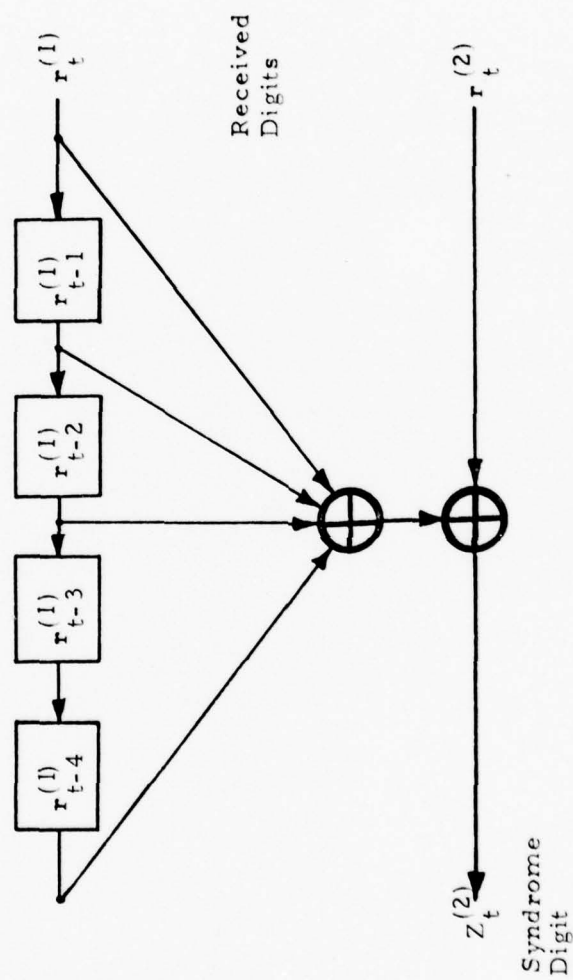
Figure 2.7a Actual Syndrome Former

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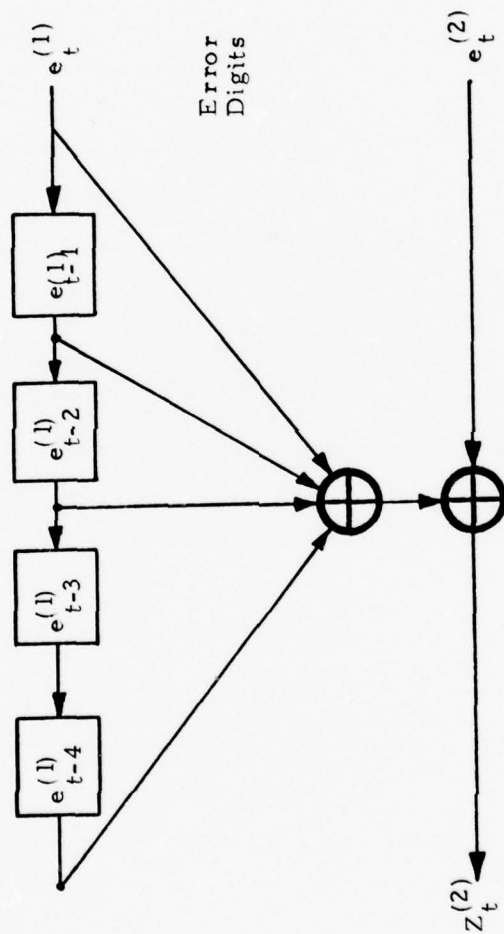
Figure 2.7b Syndrome Former Analyzed

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a syndrome vector Z as consisting of all syndromes on the left hand side of the set of syndrome equations delimited by a specified time span. Thus for the above example the syndrome vector is

$$\underline{Z} = (Z_{t-5}, Z_{t-4}, \ldots, Z_t) = Z_{(t-5, t)} = Z_{t-5}^t \qquad (2.21)$$

where four equivalent notations which will appear later are given. The number of bits in a given syndrome vector $\underline{Z}$ will be denoted as either $|Z|$ or LSYN in following work. Note that a particular span of syndrome equations such as Eq. (2.20) specifies a set of error digits on the right hand side. It will be convenient to define an overall error vector, $\underline{E}'$, as the total set of error digits appearing in such a span of equations. When later discussing single error bit estimation, the error vector, $\underline{E}$, will be used to represent the set of such error digits less the particular bit to be estimated. The number of error bits in $\underline{E}$ will be identified as LEN. Thus the number of error bits in $\underline{E}'$ will be LEN + 1.

While on the subject of syndromes, it will prove useful to later work to here define the syndrome former state, $S_t$, as being the binary vector represented by the logical values present in the physical storage elements of the syndrome former at time t. Thus one could represent the operation of the syndrome former in terms of a syndrome former state diagram similar to that defined for the encoder. For example the K = 5 rate $\frac{1}{2}$ syndrome former of Figure 2.7 could

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be represented as a state diagram having $2^\nu = 2^4$ states. The syndrome former state transitions are determined in this example by the value of $e_t^{(1)}$. Thus the _syndrome former state transition probability_ $P(S_{t+1}/S_t)$ for the two allowable transitions is just equal to $P(e_t^{(1)} = 0)$ and $P(e_t^{(1)} = 1)$ respectively.

However the _syndrome former output probability_ is more complex than in the encoder case since the output $Z_t$ depends not only on the current state $S_t$ and the input $e_t^{(1)}$ but also upon $e_t^{(2)}$ (see Figure 2.7b). This complication will be considered in detail in Chapter 3. For now it is sufficient to note that the conditional _syndrome former output probability_, $P(Z_t/S_t, S_{t+1})$ is independent of all prior syndrome former states $S_j$ for $j < t$. Therefore,

$$P(Z_t/S_0, S_1, \ldots, S_t, S_{t+1}) = P(Z_t/S_t, S_{t+1}) \tag{2.22}$$

Thus the syndrome former also displays a Markov property analogous to that of the encoder. This property will be exploited in Chapter 3 to simplify analysis of optimal decoding by error estimation using a syndrome former.

## 2.4 Decoding

Convolutional encoding has been shown in Section 2.2 to be relatively simple to implement. Decoding of convolutional codes is somewhat more difficult. The goal of a decoder is to provide to the data user an estimate of the data source output bit $I_t$ or sequence

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$\underline{I}_{(t_1, t_2)}$ . Probabilistic decoding is a method of doing so by process-
ing information available at the decoder and outputing as an estimate
that value $\hat{I}_t$ (or $\hat{\underline{I}}$) with the highest probability of occurrence con-
ditioned on the information available to the decoder.

Since the noise is additive (i.e. $\underline{R}_t = \underline{C}_t + \underline{E}_t$), there are two
complementary approaches to such estimation. In direct data estima-
tion the most probable $I_t$ (or $\underline{I}$) is determined conditioned on available
information (e.g. a received vector $\underline{R}$). An equivalent form of data
estimation first produces code digit (or code sequence) estimates
then deterministically estimates corresponding databits(or sequences)
by solving Eq. (2.4). In error estimation the most probable additive
transmission errors are estimated conditioned on available informa-
tion and then subtracted from the appropriate received data to provide
code digit $C_t$ or code sequence $\underline{C}$ estimates. With such error esti-
mates, Eq. (2.4) then provides a means of determining actual data
bit or sequence estimates. Figure 2.8 is an example of a case in
which the available information to the decoder consists of syndrome
bits which are used to form the error estimate $\hat{e}_t$ .

Several probabilistic approaches to bit and sequence decoding
are qualitatively discussed in the next two sections to show the dis-
tinction between data estimation and error estimation. Error approa-
ches explored in this work are noted.

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### 2.4.1 Data Estimation

A number of researchers have investigated optimal approaches to decoding convolutional codes by data estimation. In (1) through (4) below their goal was to estimate the encoded information bit $I_t$ or sequence $I_{(1, N)}$ by processing the entire finite received sequence $R_{(1, N + \nu)}$.

(1) Conventional Viterbi Decoding selects the information sequence $\hat{I}_{(1, N)}$ to be the information sequence which maximizes

$$P(I_{(1, N)} / R_{(1, N + \nu)})$$

where equilikely input sequences are assumed. Consequently, this approach is Maximum Likelihood Sequence Decoding. (Viterbi [5]).

(2) Map Sequence Decoding selects the information sequence $\hat{I}_{(1, N)}$ to be the information sequence which maximizes

$$P\left(I_{(1, N)} / R_{(1, N + \nu)}\right) = \frac{P\left(I_{(1, N)}, R_{(1, N + \nu)}\right)}{P\left(R_{(1, N + \nu)}\right)}$$

where nonequilikely input sequences are considered. If the input sequences are equilikely, conventional Viterbi decoding is obtained (McAdam [9]).

(3) Map Bit Decoding selects the information bit $\hat{I}_t$ to be that information bit which maximizes

$$P(I_t / R_{(1, N + \nu)}) = \frac{P(I_t, R_{(1, N + \nu)})}{P_{(R (1, N + \nu))}}$$

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(McAdam [9] .

(4) <u>Maximum</u> <u>Likelihood</u> <u>(ML)</u> <u>Bit</u> <u>Decoding</u> is the special case of map
bit decoding when information bits are equilikely.

Since it may not be possible or desirable in practice to process
the entire received sequence, as required for the optimal approaches
(1) through (4) above, some work has appeared in which a <u>finite</u> <u>stor-</u>
<u>age</u> <u>length</u> of the possibly infinite length received sequence was used
to estimate either an information bit $\hat{I}_t$ (or $\hat{I}_{t-\Delta}$) or information se-
quence $\hat{I}_{(1, t)}$. This "real time constraint" on the information bit/se-
quence estimation has been considered by the following two approaches.

(5) <u>Real</u> <u>Time</u> <u>Viterbi</u> <u>Sequence</u> <u>Decoding</u> seeks that information
sequence $\hat{I}_{(1, t+\Delta)}$ which maximizes

$$P(I_{(1, t+\Delta)} / R_{(1, t+\Delta)}) \quad \text{and sets } \hat{I}_t = \text{t-th bit of}$$
$$\text{sequence } \hat{I}_{(1, t+\Delta)}$$

This is a finite delay version of the conventional Viterbi Algorithm
for equilikely input sources (Lee [11] ) .

(6) <u>Real</u> <u>Time</u> <u>Minimum</u> <u>Bit</u> <u>Error</u> (RTMBEP) <u>Decoding</u> seeks that
information bit $\hat{I}_t$ which maximizes

$$P(I_t / R_{(1, t+\Delta)}) .$$

This is a finite delay version of McAdam Map Bit Decoding, special-
ized to equilikely input bits. (Lee [11] ) .

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These six approaches sought a particular type of estimate of the information bit $\hat{I}_t$ or sequence $\hat{I}_{(1, t)}$ derived based on the receipt and availability of some received vector or vector segment (e.g. $R_{(1, t + \Delta)}$).

### 2.4.2 Error Estimation

Instead of estimating the information bit/sequence as above, one might consider estimating <u>error</u> bits $\hat{e}_t$ or <u>sequences</u> $\hat{e}_{(1, t)}$ and obtaining an estimate of the information bits by subtracting the estimated error bits from the appropriate received bits. In such an approach, a Syndrome Former could be used to produce syndrome bits $Z_t$ from the received vector $\underline{R}_t$, thereby making available to the decoder a vector $\underline{Z}_{(1, t)}$ which is independent of the information bits or sequences actually sent and dependent only on the channel errors which have occurred. Then an optimal estimate of the error bit $\hat{e}_t$ or sequence $\hat{e}_{(1, t)}$ could be derived analogous to the approaches taken in the case of information bit/sequence estimation.

One complicating factor in such an error estimation approach is that channel errors are not equilikely (otherwise it would be senseless to try to transmit information over such a channel). Therefore, Maximum Likelihood (ML) estimation must <u>not</u> be used. Rather, a Maximum A Posteriori Probability (MAP) estimate should be implemented which takes into account the nonequilikely apriori probabilities of channel error values which are related to the $E_s/N_o$ level in the

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applicable channel.

Another source of concern in the error estimation approach would be the number of error bits or sequences to (potentially) be estimated. For example, in conventional information bit decoding of a rate 1/3 convolutional code, only an estimate of a single information bit (e.g. $I_t$) is required. In error estimation for this code, on the surface it might seem necessary to consider estimating as many as 3 error bits. It will be seen that it is not always necessary to estimate all error bits. Specifically, in a rate 1/3 systematic code it is obvious that only a single error bit estimate is required. The error estimation approaches in this work have been derived for rate 1/n systematic codes. However, the extension of such probabilistic decoding to non-systematic codes is relatively straightforward although this requires estimation of more than a single error bit.

In this investigation the error estimation approach has been pursued and the following four estimates have been derived.

(1) Map Error Bit Estimate Decoding which selects as the error bit estimate $\hat{e}_t$ that error bit value which maximizes

$$P(e_t / Z_{(1, N + \nu)}) \; .$$

This decoding approach is basically an adaptation of the approach used by McAdam [9] and Bahl et al [10] for MAP data bit estimation.

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(2) <u>Map</u> <u>Error</u> <u>Sequence</u> <u>Decoding</u> which selects that error sequence $\hat{e}_{(1, N)}$ which maximizes

$$P(e_{(1, N)} / Z_{(1, N + \nu)}).$$

This approach can be viewed in general as an adaptation of Viterbi decoding to the nonequilikely error sequence decoding case. Also pursued is a conversion of a specific rate $1/3$ systematic code error sequence decoding problem to a rate $1/2$ non-systematic decoding configuration in which the minimum Hamming weight error sequence approach of Schalkwijk and Vinck [13] is useful.

Approaches (1) and (2) assume the availability of the <u>entire</u> received sequence $R_{(1, N + \nu)}$ by assuming the availability of the entire syndrome vector $\underline{Z}_{(1, N + \nu)}$ in (1) and of essentially infinite storage in (2). These are developed in detail in Chapter 3.

Approaches (3) and (4) deal with the potentially more practical case of <u>finite</u> delay-<u>finite</u> storage (i.e. "real time") MAP error decoding.

(3) <u>Real</u> <u>Time</u> <u>MAP</u> <u>Error</u> <u>Bit</u> <u>Decoding</u> which selects that error bit value $\hat{e}_t$ which maximizes

$$P(e_t / Z_{(1, t + \Delta)}).$$

This is essentially an adaptation for error estimation of Lee's RTMBEP Decoding of information bits [11].

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(4) <u>Real</u> <u>Time</u> <u>Viterbi</u> <u>MAP</u> <u>Error</u> <u>Sequence</u> <u>Decoding</u> which selects

that error sequence $\hat{e}_{(1, \ t + \Delta)}$ which maximizes

$$P(e_{(1, \ t + \Delta)} \ / \ Z_{(1, \ t + \Delta)})$$

and selects $\hat{e}_t$ as the t-th bit of this sequence. This is basically an

adaptation for error estimation of Lee's Real Time Viterbi Decoding

·[11] Approaches (3) and (4) are developed in detail in Chapter 3.

Note that all of the above error estimation approaches have been

MAP estimates. This is due to the nonequilikely nature of the channel

errors.

After considering the above "optimal" approaches to MAP <u>error</u>

estimation, a suboptimal method of MAP error estimation has been

derived in Chapter 4 which eliminates most active computations dur-

ing decoding. This ROMMAP approach selects as the error bit esti-

mate $\hat{e}_{t - \Delta}$ that error bit value which maximizes

$$P(e_{t - \Delta} \ / \ Z_{(t - \Delta, \ t)}) \ .$$

(Basically, this is a finite delay ($\Delta$) - finite storage (L) MAP esti-

mate).

The ROMMAP is a <u>suboptimal</u> attempt at implementing finite de-

lay-finite storage ("real time") MAP Error Bit Decoding (see (3)

above) since, to obtain a readily computable ROMMAP look-up table,

that portion of the total syndrome vector $Z_{(1, \ t)}$ from initialization to

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time $t - L - 1$ (i.e. $Z_{(1, t - L - 1)}$ has been <u>neglected.</u> Consequently, the ROMMAP performance should be inferior to the optimal real time.

Figure 2.9. Convolutionally Encoded Communication System
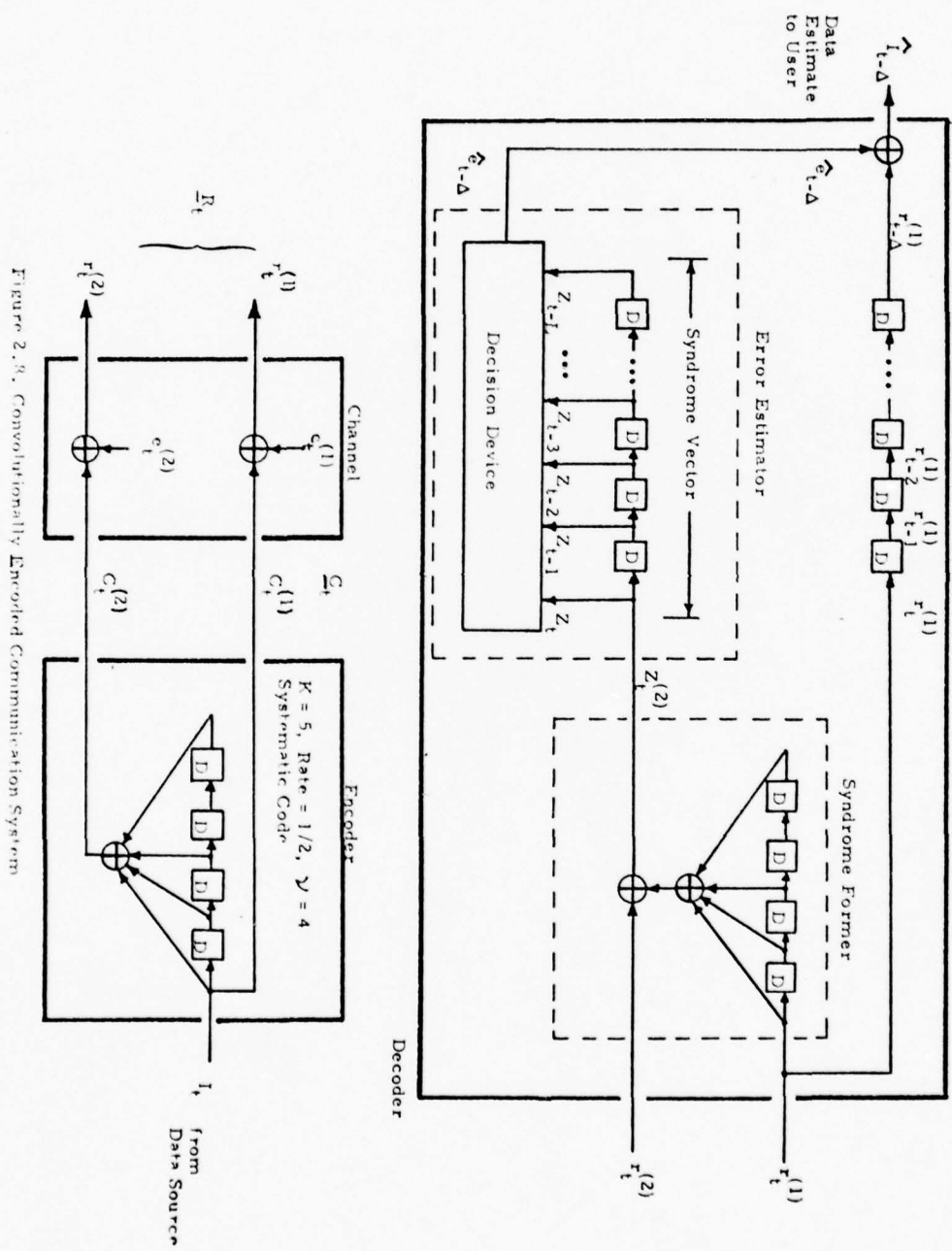
## Chapter 3

## OPTIMAL ERROR ESTIMATION

In this chapter optimal decoding of convolutional codes via error estimation is explored in some depth. Several estimation techniques previously used only for data estimation are adapted and applied to the problem of error estimation based on syndromes. First a recursive MAP error bit estimation technique is developed for the case in which the entire syndrome vector $\underline{Z}_{(1,\tau)}$ is available (i.e. no finite delay constraint exists). Next, a MAP error sequence estimator with some analogy to the Viterbi maximum likelihood sequence decoding algorithm is analyzed. Then a somewhat more realistic case of optimal MAP error bit decoding with a finite delay constraint is derived in detail. For completeness, a corresponding finite delay Viterbi MAP error sequence technique is exhibited. Finally, the relationship between MAP error bit decoding using a syndrome $\underline{Z}$ and MAP data bit decoding using the received vector $\underline{R}$ is locked at from a conceptual viewpoint. This chapter contains the theoretically optimal approaches to error *estimation which serve as the base from which* feasible suboptimal implementations of error decoders are obtained in Chapter 4.

### 3.1 Recursive MAP Error Bit Estimation

To implement a MAP estimate of an error bit $e_t$ given an

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available syndrome vector $\underline{Z}$ one would normally seek to select that value $\hat{e}_t$ of $e_t$ so as to maximize

$$P(e_t/\underline{Z}) = \frac{P(\underline{Z}/e_t)P(e_t)}{P(\underline{Z})} \qquad (3.1)$$

Suppose one attempts to do this via a recursive estimate of $e_t^{(j)}$ given the total syndrome vector $\underline{Z}_1^\tau$ produced as a result of syndrome bit computation based on the total vector of received bits $\underline{R}$ representing an input information stream of length T followed by $\nu$ zeros to drive the convolutional encoder back to the all zero state. We can do so by using the fact that the syndrome former is a linear sequential circuit whose output $Z_t$ depends only on the current syndrome former physical state $S_t$ and the incoming error bits at time t, namely $E_t$. Here $\tau = T + \nu$.

Unlike the encoder itself, the syndrome former does not necessarily store all input error bits in $E_t$. Therefore in general one cannot say that the pair of states $S_t$, $S_{t+1}$ defined by actual physical storage elements determine an output $Z_t$. For example, in the rate 1/2 systematic code example the error bit $e_t^{(2)}$ contributes to the syndrome output $Z_t$ but does not appear in any component of the physical state $S_t$ represented by the actual contents of the storage elements in the syndrome former.

Both McAdam [9] and Bahl et al. [10] derived MAP estimates of information bits given the total received vector $\underline{R}$. In the following,

the approach and notation of Bahl will be applied to the derivation of a MAP estimate of error bits $e_t^{(j)}$ given the total computed syndrome vector $\underline{Z}_1^\tau$. As usually done (e. g. Schalkwijk and Vinck [13]) we consider the syndrome former inputs to be error bits only since the syndrome former output is a function of these errors only and not of the code bits which also enter in $R_t$.

For the sake of clarity we define the following assumption and later show how although it was true in Bahl's situation it does not have to be true for the syndrome case at critical steps.

Assumption #1: The syndrome former has physical storage elements (whose contents we call $S_t$) which contain all components of $R_t$. With this assumption we can treat the syndrome former as a Markov source whose state pairs $S_t$, $S_{t+1}$ determine $Z_t$ and proceed as in Bahl's work. We will note if and when this assumption is actually required in the syndrome case.

The next figure shows a block diagram comparison between Bahl's problem and ours:
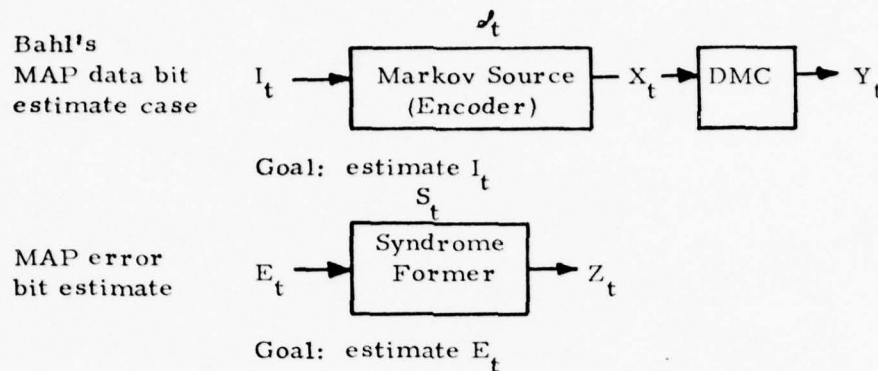
Figure 3.1. Comparison of Data and Error Estimation Approaches.

We now follow Bahl's notation and estimate the a posteriori probability of the states and transitions of a syndrome former. In the following, m is one of the $2^{\nu}$ possible binary values of a state.

Source Transition Probabilities

Syndrome former state transition probabilities may be expressed as:

$$p_t(m/m') = \Pr(S_t=m/S_{t-1}=m') \tag{3.2}$$

Our syndrome former state transition probabilities are governed by the statistics of the input n-tuple,

$$E(t) = \begin{bmatrix} e_t^{(1)} \\ e_t^{(2)} \\ \vdots \\ e_t^{(n)} \end{bmatrix} \tag{3.3}$$

through

$$Pr(S_t=m/S_{t-1}=m') = \sum_{E_t} Pr(S_t=m, E_t/S_{t-1}=m')$$

$$= \sum_{E_t} Pr(S_t=m/E_t, S_{t-1}=m')Pr(E_t/S_{t-1}=m') \tag{3.4}$$

Since $E_t$ is not a function of $S_{t-1}=m'$ we have

$$Pr(S_t=m/S_{t-1}=m') = \sum_{E_t} Pr(S_t=m/E_t, S_{t-1}=m')Pr(E_t) \tag{3.5}$$

Since the channel is memoryless, the individual bits in $E_t$ are bit-to-bit independent and one could let $Pr(E_t) = \prod_{j=1}^{n} Pr(e_t^{(j)})$. The first factor on the right is just 0 or 1 depending on whether the transition is possible and that the particular $E_t$ being considered in the sum is that $E_t$ which causes such a transition (or one of the several $E_t$ which can cause the same transition in the case of some don't care elements $e_t^{(j)}$ in $E_t$).

The don't care elements can be eliminated from the above sum by defining

$$E_t^* \triangleq (\text{those } e_t^{(j)} \text{ which actually enter into physical syndrome state storage elements})$$

yielding:

$$Pr(S_t=m/S_{t-1}=m') = \sum_{E_t^*} Pr(S_t=m/E_t^*, S_{t-1}=m')Pr(E_t^*) . \tag{3.6}$$

## Output Probabilities

Syndrome former output probabilities may be written as:

$$q_t(Z/m', m) = Pr(Z_t = Z/S_{t-1} = m', S_t = m). \qquad (3.7)$$

Note that $q_t(Z/m', m) = 0$ or $1$ when Assumption #1 is true since $S_{t-1}$, $S_t$ together contain exactly those error bits needed to determine $Z_t$. If Assumption #1 is not true, then $q_t(Z/m', m)$ would not necessarily be 0 or 1 but would be dependent on other error bits which never reside in the physical state. However in such a case one can average over these other error bits to still obtain:

$$q_t(Z/m', m) = \sum_{\delta_t} Pr(Z_t = Z/S_{t-1} = m', S_t = m, \delta_t = \delta) Pr(\delta_t = \delta) \qquad (3.8)$$

where $\delta_t \triangleq$ all those error bits in $E_t$ which affect the syndrome $Z_t$ but which are not present in the actual physical state $S_{t-1}$, $S_t$.

## Objective of the Decoder

The objective of the decoder is to examine $Z_1^\tau$ and estimate the a posteriori probability of the syndrome former _states_:

$$Pr(S_t = m/Z_1^\tau) = \frac{Pr(S_t = m, Z_1^\tau)}{Pr(Z_1^\tau)} \triangleq \frac{\lambda_t(m)}{Pr(Z_1^\tau)} \qquad (3.9)$$

and _transitions_

$$Pr(S_{t-1} = m', S_t = m/Z_1^\tau) = \frac{Pr(S_{t-1} = m', S_t = m, Z_1^\tau)}{Pr(Z_1^\tau)} \triangleq \frac{\sigma_t(m', m)}{Pr(Z_1^\tau)}. \qquad (3.10)$$

Now derive the joint probabilities defined above, namely,

$$\lambda_t(m) \overset{\Delta}{=} Pr(S_t = m, Z_1^\tau) \quad \text{and} \tag{3.11a}$$

$$\sigma_t(m', m) = Pr(S_{t-1} = m', S_t = m, Z_1^\tau) . \tag{3.11b}$$

To simplify notation, define

$$\alpha_t(m) \overset{\Delta}{=} Pr(S_t = m, Z_1^t) \tag{3.12a}$$

$$\beta_t(m) \overset{\Delta}{=} Pr(Z_{t+1}^\tau / S_t = m) \tag{3.12b}$$

$$\gamma_t(m', m) \overset{\Delta}{=} Pr(S_t = m, Z_t / S_{t-1} = m') \tag{3.12c}$$

Now

$$\lambda_t(m) = Pr(S_t = m, Z_1^\tau) = Pr(S_t = m, Z_1^t, Z_{t+1}^\tau)$$

$$= Pr(S_t = m, Z_1^t) Pr(Z_{t+1}^\tau / S_t = m, Z_1^t) \tag{3.13}$$

Note that the statistics of $Z_{t+1}^\tau$ are not dependent on past syndrome bits when the condition on $S_t$ is applied. For example $Z_{t+1}^\tau$ is a function of $S_t$ and $E_{t+1}$ but $Z_1^t$ doesn't provide any additional information. Therefore,

$$\lambda_t(m) = Pr(S_t = m, Z_1^t) Pr(Z_{t+1}^\tau / S_t = m)$$

$$\lambda_t(m) = \alpha_t(m) \beta_t(m) \tag{3.14}$$

Similarly,

$$\sigma_t(m', m) = Pr(S_{t-1} = m', S_t = m, Z_1^\tau) = Pr(S_{t-1} = m', S_t = m, Z_1^{t-1}, Z_t, Z_{t+1}^\tau)$$

$$= Pr(S_{t-1} = m', Z_1^{t-1}) Pr(S_t = m, Z_t / S_{t-1} = m', Z_1^{t-1})$$

$$\cdot Pr(Z_{t+1}^\tau / S_{t-1} = m', Z_1^{t-1}, S_t = m, Z_t) \tag{3.15}$$

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Here use has been made of the fact that:

(1) $S_t=m$, $Z_t$ are dependent on $S_{t-1}=m'$ but <u>not</u> on $Z_1^{t-1}$ (past syndrome bits) when conditioned on $S_{t-1}=m'$; and

(2) $Z_{t+1}^{\tau}$ when conditioned on $S_t=m$ is not statistically dependent on $S_{t-1}=m'$ (past state), $Z_1^{t-1}$ (past syndrome bits), or $Z_t$ (immeidately past syndrome bits).

Therefore we can write the joint probability as

$$\sigma_t(m',m) = Pr(S_{t-1}=m', Z_1^{t-1})Pr(S_t=m, Z_t/S_{t-1}=m')Pr(Z_{t+1}^{\tau}/S_t=m)$$

$$\sigma_t(m',m) = \alpha_{t-1}(m')\gamma_t(m',m)\beta_t(m) \tag{3.16}$$

<u>Forward Recursion</u>

Now for $t = 1, 2, 3, \ldots, \tau$ a forward recursion can be derived as follows.

$$\alpha_t(m) = Pr(S_t=m, Z_1^t) = \sum_{\text{all } S_{t-1}=m'} Pr(S_{t-1}=m', S_t=m, Z_1^t)$$

$$= \sum_{\text{all } S_{t-1}=m'} Pr(S_{t-1}=m', S_t=m, Z_1^{t-1}, Z_t)$$

$$= \sum_{\text{all } S_{t-1}} Pr(S_{t-1}=m', Z_1^{t-1})Pr(S_t=m, Z_t/S_{t-1}=m', Z_1^{t-1}) \tag{3.17}$$

Using fact (1) above one obtains:

$$\alpha_t(m) = \sum_{\text{all } S_{t-1}} Pr(S_{t-1}=m', Z_1^{t-1})Pr(S_t=m, Z_t/S_{t-1}=m')$$

$$\alpha_t(m) = \sum_{\text{all } S_{t-1}} \alpha_{t-1}(m')\gamma_t(m', m) \qquad (3.18)$$

At $t=0$ we have boundary conditions: $\alpha_0(0) = 1$, $\alpha_0(m) = 0$ for all $m \neq 0$ since the syndrome former $S_0$ state is $\underline{known}$ to be $\underline{0}$.

## Backward Recursion

In a similar manner the backward recursion may be obtained as follows:

$$\beta_t(m) = Pr(Z_{t+1}^\tau/S_t=m) = \sum_{\text{all } S_{t+1}} Pr(S_{t+1}=m', Z_{t+1}^\tau/S_t=m)$$

$$= \sum_{\text{all } S_{t+1}} Pr(S_{t+1}=m', Z_{t+1}, Z_{t+2}^\tau/S_t=m)$$

$$= \sum_{\text{all } S_{t+1}} Pr(S_{t+1}=m', Z_{t+1}/S_t=m)Pr(Z_{t+2}^\tau/S_{t+1}=m', Z_{t+1}, S_t=m)$$

$$(3.19)$$

Again we can state that $Z_{t+2}^\tau$ when conditioned on $S_{t+1}=m'$ is statistically independent of $Z_{t+1}$ (past syndrome bits) and of $S_t=m$ (past syndrome former state).

Therefore,

$$\beta_t(m) = \sum_{\text{all } S_{t+1}} \beta_{t+1}(m')\gamma_{t+1}(m, m')$$

$$(3.20)$$

with boundary conditions $\beta_\tau(0) = 1$, $\beta_\tau(m) = 0$ for all $m \neq 0$ since we have assumed that we drive the syndrome former to $\underline{0}$ state once we reach the point in time when it is known that only the tail of $\nu$ zero bits are being encoded at the transmitter. Now

$$\gamma_t(m', m) \overset{\Delta}{=} \Pr(S_t = m, Z_t / S_{t-1} = m')$$

$$= \Pr(S_t = m / S_{t-1} = m')\Pr(Z_t / S_{t-1} = m', S_t = m)$$

$$\gamma_t(m', m) = p_t(m/m')q_t(Z_t/m', m) \tag{3.21}$$

Note that we do $\underline{not}$ have to average over any intermediate X since in the syndrome case we have $\underline{no}$ $X_t$ as was present in Bahl's problem situation.

Operation of Decoder

The decoder must compute the following.

$$\lambda_t(m) = \Pr(S_t = m, Z_1^\tau) = \alpha_t(m)\beta_t(m) \qquad \text{Syndrome Former State Joint Probability} \tag{3.22}$$

$$\sigma_t(m', m) = \Pr(S_{t-1} = m', S_t = m, Z_1^\tau) = \sum_{S_{t-1} = m'} \alpha_{t-1}(m')\gamma_t(m', m)$$

State Transition Joint Probability $\tag{3.23}$

These computations may be accomplished in the following steps:

Step #1:

$\alpha_0(m)$, $\beta_\tau(m)$ $m = 0, 1, \ldots, M-1$ are initialized $(M = 2^\nu)$.

$$\alpha_0(0)=1, \alpha_0(m)=0 \text{ for all } m\neq0$$
$$\beta_\tau(0)=1, \beta_\tau(m)=0 \text{ for all } m\neq0$$

$\left.\vphantom{\begin{array}{c}a\\b\end{array}}\right\}$ Assuming Initial and Final States <u>Known</u>

$(3.24)$

<u>Step #2:</u>

As soon as each $Z_t$ is obtained compute

$$\gamma_t(m', m) = p_t(m/m')q_t(Z_t/m', m) = Pr(S_t=m, Z_t/S_{t-1}=m') \tag{3.25}$$

Next compute

$$\alpha_t(m) = \sum_{S_{t-1}=m'} \alpha_{t-1}(m')\gamma_t(m', m) \tag{3.26}$$

<u>Step #3:</u>

<u>After</u> the complete sequence $Z_1^\tau$ has been received, recursively compute

$$\hat\beta_t(m) = \sum_{m'} \hat\beta_{t+1}(m')\gamma_{t+1}(m, m') \tag{3.27}$$

and multiply by the stored $\alpha_t(m)$ to obtain

$$\lambda_t(m) = \alpha_t(m)\beta_t(m) \tag{3.28}$$

and by $\gamma_t(m', m)$ and $\alpha_{t-1}(m')$ to obtain

$$\sigma_t(m', m) = \alpha_{t-1}(m')\gamma_t(m', m)\beta_t(m). \tag{3.29}$$

<u>Error</u> Bit Estimation

Having determined $\lambda_t(m)$ and $\sigma_t(m', m)$ for all $0 \le t \le \tau$, now estimate the required error bits $\{e_t^{(j)}\}$. Assume that the error

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bits required to appear explicitly in the syndrome former physical state.

Let $A_t^{(j)}$ be the set of syndrome former states $S_t$ such that $s_t^{(j)} = 0$ where $s_t^{(j)}$ is the value of $e_t^{(j)}$ after it is shifted into the associated syndrome former storage element.

Then $s_t^{(j)} = e_t^{(j)}$ for $e_t^{(j)} \epsilon E^*$ where $E^* \overset{\Delta}{=}$ [those $e_t^{(j)}$ which must be estimated and appear explicitly in syndrome former state]. This implies

$$Pr(e_t^{(j)} = 0, Z_1^\tau) = \sum_{S_t \epsilon A_t^{(j)}} \lambda_t(m) . \qquad (3.30)$$

Normalizing by

$$Pr(Z_1^\tau) = \lambda_\tau(0) , \qquad (3.31)$$

the conditional probability,

$$Pr(e_t^{(j)} = 0/Z_1^\tau) = \frac{1}{\lambda_\tau(0)} \sum_{S_t \epsilon A_t^{(j)}} \lambda_t(m), \qquad (3.32)$$

is obtained. Now <u>decode the error bit</u> by the rule:

$$\hat{e}_t^{(j)} = 0 \text{ if } Pr(e_t^{(j)} = 0/Z_1^\tau) \geq 0.5 \qquad (3.33)$$

$$\hat{e}_t^{(j)} = 1 \quad \text{otherwise.}$$

Thus the MAP estimate of a desired error bit $e_t^{(j)}$ given the entire syndrome vector $Z_1^\tau$ has been obtained on a recursive basis.

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It is seen that Assumption #1 is not needed for this error bit esti-
mation as long as the error bit to be estimated does appear in the
syndrome former physical state.

## 3.2 Syndrome Decoding of Rate 1/3 Systematic Codes

In the case of rate $1/n$ systematic convolutional codes the
syndrome based decoder does not have to estimate all error bits in

$$E_t = \begin{bmatrix} e_t^{(1)} \\ e_t^{(2)} \\ \vdots \\ e_t^{(n)} \end{bmatrix} \tag{3.34}$$

The other error bits $e_t^{(2)}, e_t^{(3)}, \ldots, e_t^{(n)}$ can be viewed as nuisance
parameters which in MAP estimation should be averaged out in the
computation of $\hat{e}_t^{(1)}$. This was the approach followed in the previous
work on recursive MAP error bit estimation.

A different approach to the decoding problem exists in which
estimation of the most probable error sequence is the goal of the
decoder. Recall that classical Viterbi Maximum Likelihood data
sequence decoding assumes equilikely input data sequences. Since
errors are very nonequilikely, Appendix A has been provided to show
in general how the basic Viterbi sequence decoding metrics would
have to be modified to account for nonequilikely inputs such as error
sequences.

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Schalkwijk and Vinck [13] have derived and demonstrated a recursive algorithm like Viterbi's which (for rate 1/2 non-systematic codes) finds the error sequence of minimum Hamming weight which could have caused the observed syndrome sequence. This minimum weight error sequence is the most probable error sequence given the observed syndrome sequence. Considering the work that has been done [13] in obtaining minimum weight error sequences, the discussion that follows merely shows how a specific rate 1/3 systematic code error sequence decoding problem may be converted to a rate 1/2 non-systematic error sequence decoding problem suitable for solution by the method of Schalkwijk and Vinck. The rate 1/3 systematic code of this example was selected from Bussgang's table of optimum subgenerators of a canonical subgenerator (Fig. 7 in [19]).

Consider the syndrome former for a rate 1/3 systematic code shown in Figure 3.2a. The goal here is to estimate the error sequence $(e_0^{(1)}, e_1^{(1)}, \ldots, e_{N-1}^{(1)})$ based on the syndrome sequences $s_{(0, N-1)}^{(2)}$ and $s_{(0, N-1)}^{(3)}$ where N is the length of the entire output sequence from the data source. Figure 3.2a can be redrawn as in Figure 3.2b to more clearly isolate the "nuisance" errors $e_t^{(2)}$ and $e_t^{(3)}$. One can thus observe that the problem of estimating the desired error sequence $e_0^{(1)}, \ldots, e_{N-1}^{(1)}$ is equivalent to estimating the "data" input sequence to a rate 1/2 convolutional "error encoder" of

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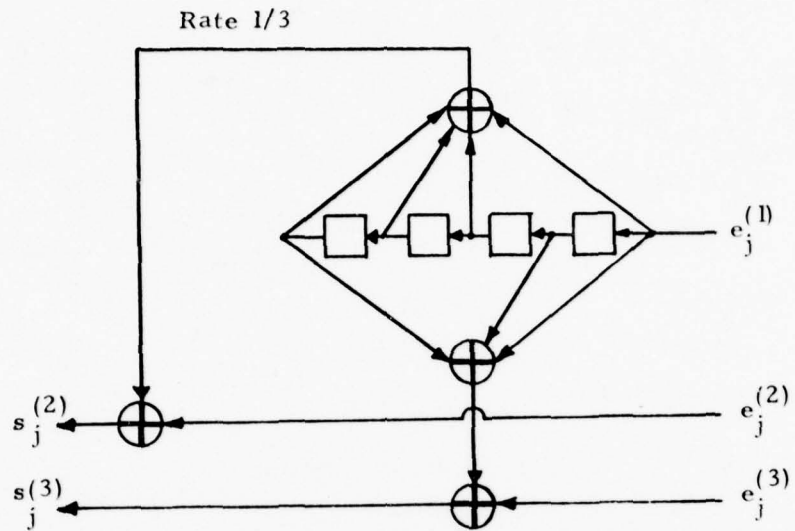
Figure 3.2a "First Syndrome Former" with Error Inputs
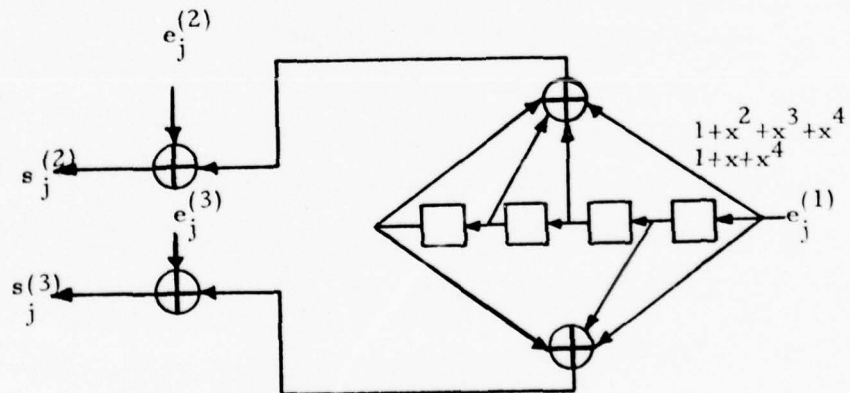


Figure 3.2b Alternate View of Syndrome Former

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Figure 3.2b. Although this appears to have reduced the rate 1/3

problem to a possibly more manageable rate 1/2 one, normal Viterbi

sequence decoding cannot be used to estimate the error input. The

classical Viterbi algorithm is a maximum likelihood decoder for

equilikely input sequences while here the input is a very non-

equilikely sequence of error bits $e_t^{(1)}$.

Now instead of pursuing a modified Viterbi algorithm with its

more complicated metric calculations for nonequilikely error input

sequences (see Appendix A), suppose error sequence decoding using

the method of Schalkwijk and Vinck [13] is followed for the case

shown in Figure 3.2b.

To keep things straight, call the units of Fig. 3.2a and 3.2b

the First Syndrome Former (error encoder). For this error encoder

there exists a corresponding Second Syndrome Former shown in

Fig. 3.3a.

The operation of this Second Syndrome Former on input

syndrome bits $s_t^{(2)}$ and $s_t^{(3)}$ can be viewed as the equivalent operation

on error bits $e_t^{(2)}$ and $e_t^{(3)}$ as shown in Fig. 3.3b. The rate 1/3

problem has thus been converted to a rate 1/2 problem exactly

like that of Schalkwijk and Vinck [13]. The minimum weight

$(e^{(2)}(x), e^{(3)}(x))$ sequence can be found by their technique and a non-

probabilistic method used to obtain the error sequence estimate for

$e_t^{(1)}$. The Schalkwijk and Vinck method is not affected by the equi-

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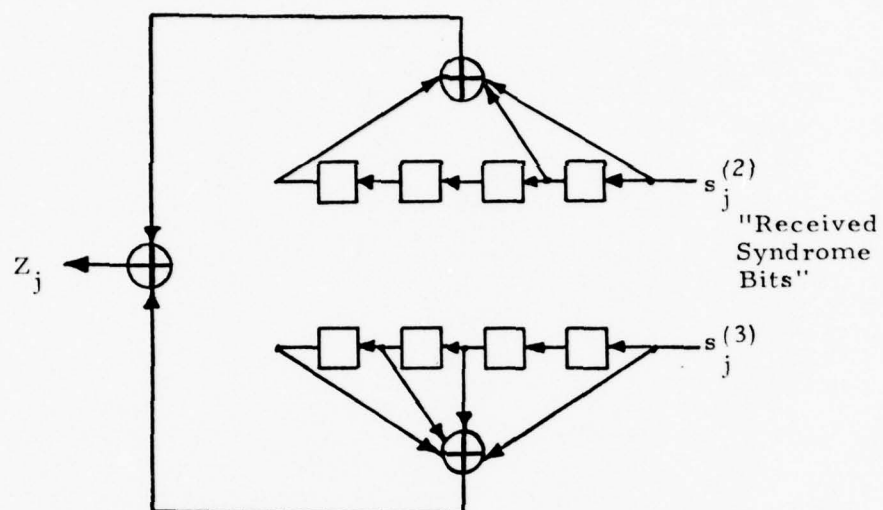
Figure 3.3a  "Second Syndrome Former" with Syndrome
Bit Inputs
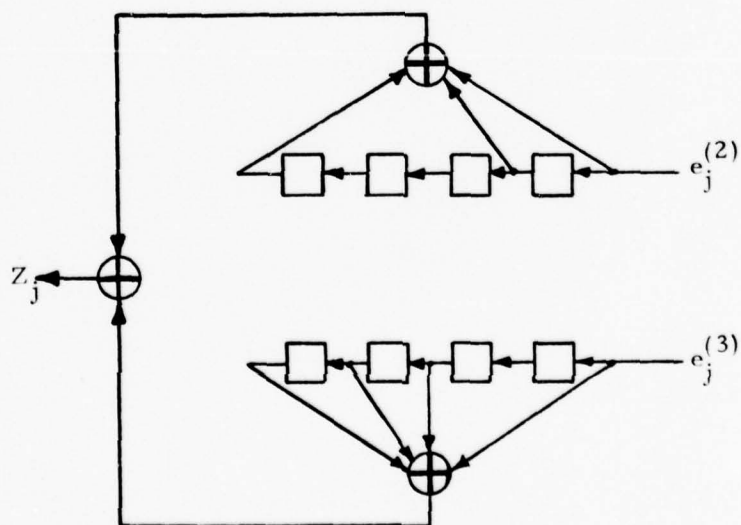


Figure 3.3b  "Second Syndrome Former" with Error Bit
Inputs

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probable or nonequiprobable nature of the encoder input $e_t^{(1)}$ since their method determines the <u>minimum</u> Hamming weight noise sequence input which due to our assumption of a BSC is the most probable sequence.

The following parallels Schalkwijk and Vinck's explanation of their approach as modified for the rate 1/3 to rate 1/2 case.

The Second Syndrome Former Output is:

$$Z(x) = C_2(x)e^{(3)}(x) + C_3(x)e^{(2)}(x) \qquad (3.35)$$

where

$$C_2(x) = 1+x^2+x^3+x^4 \qquad (3.36a)$$

$$C_3(x) = 1+x+x^4 \qquad (3.36b)$$

Here x plays the role of the delay operator.

By Euclid's algorithm there exists $D_2(x)$ and $D_3(x)$ such that

$$D_2(x)C_2(x) + D_3(x)C_3(x) = 1 \qquad (3.37)$$

yielding

$$D_2(x) = x + 1 \qquad (3.38a)$$

$$D_3(x) = x. \qquad (3.38b)$$

Received are

$$s^{(2)}(x) = C_2(x)e^{(1)}(x) + e^{(2)}(x) \qquad (3.39a)$$

$$s^{(3)}(x) = C_3(x)e^{(1)}(x) + e^{(3)}(x) . \qquad (3.39b)$$

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The error sequence _estimate_ is formed as

$$\hat{e}^{(1)}(x) = D_2(x)(s^{(2)}(x) + \hat{e}^{(2)}(x)) + D_3(x)(s^{(3)}(x) + \hat{e}^{(3)}(x)). \quad (3.40)$$

If the noise sequence estimate $(\hat{e}^{(2)}(x), \hat{e}^{(3)}(x))$ is correct then

$$\hat{e}^{(1)}(x) = D_2(x)(C_2(x)e^{(1)}(x) + e^{(2)}(x) + \hat{e}^{(2)}(x))$$

$$+ D_3(x)(C_3(x)e^{(1)}(x) + e^{(3)}(x) + \hat{e}^{(3)}(x))$$

$$= (D_2(x)C_2(x) + D_3(x)C_3(x))e^{(1)}(x) = e^{(1)}(x). \quad (3.41)$$

Equation (3.40) can be written as

$$\hat{e}^{(1)}(x) = (D_2(x)s^{(2)}(x) + D_3(x)s^{(3)}(x)) + \underbrace{D_2(x)\hat{e}^{(2)}(x) + D_3(x)\hat{e}^{(3)}(x)}_{\hat{w}(x)}$$

$$(3.42)$$

where with

$$w(x) = D_2(x)e^{(2)}(x) + D_3(x)e^{(3)}(x) \quad (3.43a)$$

$$\hat{w}(x) = D_2(x)\hat{e}^{(2)}(x) + D_3(x)\hat{e}^{(3)}(x) \quad (3.43b)$$

one obtains the following:

$$\hat{e}^{(1)}(x) = \underbrace{(D_2(x)s^{(2)}(x) + D_3(x)s^{(3)}(x))}_{\substack{\text{Computed easily from the} \\ \text{received } s^{(2)}(x) \text{ and } s^{(3)}(x)}} + \underbrace{\hat{w}(x)}_{\substack{\text{Determined via} \\ \text{Viterbi-like} \\ \text{algorithm}}} \quad (3.44)$$

Since $\hat{w}(x)$ can be determined based only on the minimum Hamming weight noise $(\hat{e}^{(2)}(x), \hat{e}^{(3)}(x))$ inputs to the Second Syndrome Former and not the data inputs $e_t^{(1)}$ to the First Syndrome Former, this approach does not have to be adjusted for nonequiprobable $e_t^{(1)}$ inputs. Once

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the $\{e_t^{(1)}\}$ sequence has been determined we only have to mod 2 add it

to the $\{r_t^{(1)}\}$ sequence to determine $\{I_t\}$.

It is interesting to note that the rate 1/3 systematic code with

$$\begin{bmatrix} G_1 & = & 10000 \\ G_2 & = & 10111 \\ G_3 & = & 11001 \end{bmatrix}^{*}$$

converts by the above Second Syndrome Former technique to the

rate 1/2 nonsystematic code $\begin{bmatrix} G_2 & = & 10111 \\ G_3 & = & 11001 \end{bmatrix}$ given in Schalkwijk and

Vinck Table III, "Codes Realized." (The code is listed as code 4

with $\begin{bmatrix} g_2 & = & 10011 \\ g_3 & = & 11101 \end{bmatrix}^{**}$ ). [13].

From the discussion in Schalkwijk and Vinck regarding Table III

is appears that had a rate 1/3 code been used with

$$\begin{bmatrix} G_1 & = & 10000 \\ G_2 & = & 11001 \\ G_3 & = & 11101 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} G_1 & = & 10000 \\ G_2 & = & 11001 \\ G_3 & = & 11011 \end{bmatrix}$$

the decoding problem could have "converted" to a rate 1/2 code

(Codes 3 and 2 respectively in Table III [13] ) whose decoding would

have required far less complexity than the Code 4 obtained from the

rate 1/3 code of this example.

*0-th order left.

**0-th order right.

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## 3.3 Optimal Real Time MAP Error Bit Estimation

In this section a recursive method for the computation of the optimum MAP error bit estimate of $e_t^{(1)}$ is derived assuming a finite delay $\Delta$. The notation used throughout this section parallels that of Fritchman and Mixsell [20] in their work on optimal real time data bit estimation.

Consider a rate $1/n$ systematic code. The task is to estimate one error bit (e.g. $e_{t-\Delta}^{(1)}$) given the output of the syndrome former from the start of the reception (i.e. $t=1$) to the present time $t=t$ allowing only a finite delay $\Delta$. That is, find the maximum a posteriori estimate of $e_{t-\Delta}^{(1)}$ given $\underline{Z}_{(1,t)}$. This can be written as $\hat{e}_{t-\Delta}^{(1)}$ where $\hat{e}_{t-\Delta}^{(1)}$ is that value of $e_{t-\Delta}^{(1)}$ that produces the

$$
\begin{matrix} \text{maximum} \\ \text{over all} \\ e_{t-\Delta}^{(1)} \text{ values} \end{matrix} \left\{ \Pr(e_{t-\Delta}^{(1)}/\underline{Z}_{(1,t)}) \right\} = \begin{matrix} \text{maximum} \\ \text{over all} \\ e_{t-\Delta}^{(1)} \text{ values} \end{matrix} \left\{ \frac{\Pr(e_{t-\Delta}, \underline{Z}_{(1,t)})}{\Pr(\underline{Z}_{(1,t)})} \right\}
$$

$$\tag{3.45}$$

where

(1) $e_{t-\Delta}^{(1)}$ is the error bit to be estimated;

(2) $\{e_t^{(1)}\}$ are error bits which <u>do</u> enter the syndrome former physical state;

(3) $\underline{Z}_{(1,t)}$ is the syndrome vector from startup to present (containing all the information possibly available to a syndrome type decoder); and

(4) $\Delta$ is the "delay" permitted before an error bit decision must be output from the decoder.

One should note that in (3.45) the estimate will be made based on the entire vector $\underline{Z}_{(1, t)}$ via a recursive technique. This is in contrast to the ROMMAP approach to be discussed later in Section 4.1, in which the estimate will be based on only the finite length syndrome segment $\underline{Z}_{(t-\Delta, t)}$ namely,

$$
\begin{matrix} \text{maximum} \\ \text{over all} \\ e_{t-\Delta}^{(1)} \text{ values} \end{matrix} \left\{ Pr(e_{t-\Delta}^{(1)} / \underline{Z}_{(t-\Delta, t)}) \right\} = \begin{matrix} \text{maximum} \\ \text{over all} \\ e_{t-\Delta}^{(1)} \text{ values} \end{matrix} \left\{ \frac{Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(t-\Delta, t)})}{Pr(\underline{Z}_{(t-\Delta, t)})} \right\}
$$

$$(3.46)$$

For now, the problem presented by Eq. ( 3.45 ) will be considered. There exist several methods of expressing the numerator joint probability $Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1, t)})$ which have been used by various authors [11], [20] in estimating information bits.

For example,

$$
Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1, t)}) = \sum_{e_{t-\Delta+1}^{(1)}} \cdots \sum_{e_t^{(1)}} Pr(e_{(t-\Delta, t)}^{(1)}, \underline{Z}_{(1, t)})
$$

$$(3.47)$$

with $e_{(t-\Delta, t)}^{(1)} = (e_{t-\Delta}^{(1)}, e_{t-\Delta+1}^{(1)}, \ldots, e_{t-1}^{(1)}, e_t^{(1)})$. Here averages are taken over all possible paths (error bit sequences) of length $\Delta$ after the particular error bit to be estimated.

By conditioning the numerator on the syndrome former state $S_{t+1}$ and averaging over all possible values of $S_{t+1}$ one obtains an

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values of $e_{t-\Delta}^{(1)}$, one can also determine a measure of decision quality by summing (in the binary case) the two numerator terms and using:

$$\left\{\begin{array}{l}\text{Decision}\\\text{quality on}\\\hat{e}_{t-\Delta}^{(1)}\end{array}\right\} = \frac{Pr(\hat{e}_{t-\Delta}^{(1)}, \underline{Z}_{(1,t)})}{Pr(\hat{e}_{t-\Delta}^{(1)}, \underline{Z}_{(1,t)}) + Pr(\overline{\hat{e}}_{t-\Delta}^{(1)}, \underline{Z}_{(1,t)})} \tag{3.50}$$

Therefore since either approach can be useful in soft decision decoding, one can somewhat arbitrarily use ( 1 ) above (similar to Fritchman and Mixsell [20] ) since this entails slightly less computations and is somewhat easier to derive than the approach (2) analogous to Lee [11] and [12].

In the following, the case of $\Delta > \nu$ is considered. This would appear to be of most applicability since one would typically use a delay $\Delta$ greater than a constraint length $(\nu + 1)$ in order to obtain as much coding gain as possible with a particular convolutional code.

For completeness, the case $\Delta \leq \nu$ is considered in Appendix C where a decision directed approach is also described based on incorporation of prior estimates of $\hat{e}_{t-\Delta-1}^{(1)}$. This is similar to Abend and Fritchman [21] equation 21b.

Here let us consider the numerator of (3.3-1)

$$Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1,t)}) = \sum_{S_{t+1}=s} Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1,t)}, S_{t+1}=s) \tag{3.51}$$

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Since $S_t = (e_{t-1}^{(1)}, e_{t-2}^{(1)}, \ldots, e_{t-\nu}^{(1)})$ and $S_{t+1} = (e_t^{(1)}, e_{t-1}^{(1)}, \ldots, e_{t-\nu+1}^{(1)})$ for this rate $1/n$ systematic code (where $\{e_t^{(1)}\}$ are those error bits entering into the syndrome former physical state), we can also write:

$$\Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1,t)}) = \sum_{e_{t-\nu+1}^{(1)}} \cdots \sum_{e_t^{(1)}} \Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1,t)}, e_{t-\nu+1}^{(1)}, \ldots, e_t^{(1)})$$

(3.52)

or

$$\Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1,t)}) = \sum_{e_{t-\nu+1}^{(1)}} \cdots \sum_{e_t^{(1)}} \Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1,t)}, e_{(t-\nu+1,t)}^{(1)})$$

(3.53)

Now consider the $\Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1,t)}, e_{(t-\nu+1,t)}^{(1)})$ terms of the numerator summation conditioned on $e_{t-\nu}^{(1)}$ and averaged over $e_{t-\nu}^{(1)}$.

$$\Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1,t)}, e_{t-\nu+1,t}^{(1)}) = \sum_{e_{t-\nu}^{(1)}} \Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1,t-1)}, \underline{Z}_t, e_{(t-\nu,t)}^{(1)})$$

$$= \sum_{e_{t-\nu}^{(1)}} \Pr(\underline{Z}_t / e_{t-\Delta}^{(1)}, \underline{Z}_{(1,t-1)}, e_{(t-\nu,t)}^{(1)}) \Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1,t-1)}, e_{(t-\nu,t)}^{(1)})$$

(3.54)

Since the statistics of $\underline{Z}_t$, when conditioned on $e_{(t-\nu,t)}^{(1)}$, are <u>not</u> a function of prior values of $\underline{Z}_{(1,t-1)}$ and since the statistics of $\underline{Z}_t$ when conditioned on $e_{(t-\nu,t)}^{(1)}$ are not a function of $e_{t-\Delta}^{(1)}$, one can write

$$\Pr(\underline{Z}_t / e_{t-\Delta}^{(1)}, \underline{Z}_{(1,t-1)}, e_{(t-\nu,t)}^{(1)}) = \Pr(\underline{Z}_t / e_{(t-\nu,t)}^{(1)}).$$

(3.55)

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Also, due to the bit-by-bit independence of the errors, one can write

$$Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1, t-1)}, e_{(t-\nu, t)}^{(1)}) =$$

$$Pr(e_t^{(1)}) Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1, t-1)}, e_{(t-\nu, t-1)}^{(1)}/e_t^{(1)}) \qquad (3.56)$$

But the statistics of $\underline{Z}_{(1, t-1)}$ and of $e_{(t-\nu, t-1)}^{(1)}$ are not a function of a "future" $e_t^{(1)}$. Therefore

$$Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1, t-1)}, e_{(t-\nu, t)}^{(1)}) = Pr(e_r^{(1)}) Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1, t-1)}, e_{(t-\nu, t-1)}^{(1)})$$

$$(3.57)$$

and one can thus write

$$Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1, t)}, e_{(t-\nu+1, t)}^{(1)}) =$$

$$Pr(e_t^{(1)}) Pr(\underline{Z}_t/e_{(t-\nu, t)}^{(1)}) Pr(e_{t-\Delta}^{(1)}, Z_{(1, t-1)} e_{(t-\nu, t-1)}^{(1)}) \qquad (3.58)$$

$$e_{t-\nu}^{(1)}$$

This is a computable recursion since:

(1) $Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1, t-1)}, e_{(t-\nu, t-1)}^{(1)})$ is the prior numerator term for estimating $e_{t-\Delta}^{(1)}$ from $\underline{Z}_{(1, t-1)}$;

(2) $Pr(e_t^{(1)})$ is known for the binary symmetric channel; and

(3) $Pr(\underline{Z}_t/e_{(t-\nu, t)}^{(1)}) = Pr(\underline{Z}_t/S_t, S_{t+1}) =$ syndrome former output probability.

Since the syndrome former output $\underline{Z}_t$ is a function of both the state pair $(S_t, S_{t+1})$ and the other "don't care" error bit(s) (i.e. those that do not enter into the syndrome former storage elements),

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one can obtain the syndrome former output probabilities by conditioning on $\mathscr{S}_t$ and averaging over $\mathscr{S}_t$ where

$$\mathscr{S}_t = \left\{ \begin{array}{l} \text{All those error bits } e_t^{(i)} \text{ in } \underline{E}_t \text{ which affect the syndrome} \\ \text{output } \underline{Z}_t \text{ but which are not present in the actual storage} \\ \text{elements representing } (S_t, S_{t+1}). \end{array} \right\} .$$

Therefore

$$\Pr(\underline{Z}_t / e_{(t-\nu, t)}^{(1)}) = \Pr(\underline{Z}_t / S_t, S_{t+1}) = \sum_{\mathscr{S}_t} \Pr(Z_t / S_t, S_{t+1}, \mathscr{S}_t) \Pr(\mathscr{S}_t). \tag{3.59}$$

The expression ( 3.53) for the numerator now becomes

$$\Pr(e_{(t-\Delta)}^{(1)}, \underline{Z}_{(1, t)}) = \sum_{e_{t-\nu+1}^{(1)}} \cdots \sum_{e_t^{(1)}} \Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1, t)}, e_{(t-\nu+1, t)}^{(1)})$$

$$= \sum_{e_{t-\nu+1}^{(1)}} \cdots \sum_{e_t^{(1)}} \sum_{e_{t-1}^{(1)}} \Pr(e_t^{(1)}) \Pr(\underline{Z}_t / e_{(t-\nu, t)}^{(1)}) \cdot$$

$$\cdot \Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1, t-1)}, e_{(t-\nu, t-1)}^{(1)}). \tag{3.60}$$

For notational convenience in the recursion explanation $g_i$ is defined as

$$g_i \overset{\Delta}{=} \Pr(e_{t-\Delta}^{(1)}, \underline{Z}_{(1, t-\Delta+i)}, e_{(t-\Delta-\nu+1+i, t-\Delta+i)}^{(1)}) \tag{3.61}$$

By the conditioning and averaging discussed above, this is equivalent to

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$$g_i = \sum_{\substack{e^{(1)}_{t-\Delta+i-\nu}}} Pr(e^{(1)}_{t-\Delta+i}) Pr(\underline{Z}_{t-\Delta+i} / e^{(1)}_{(t-\Delta-\nu+i,\, t-\Delta+i)})$$

$$\cdot Pr(e^{(1)}_{t-\Delta}, \underline{Z}_{(1,\, t-\Delta+i-1)}, e^{(1)}_{(t-\Delta-\nu+i,\, t-\Delta+i-1)}) \tag{3.62}$$

where

$$g_0 = Pr(e^{(1)}_{t-\Delta}, \underline{Z}_{(1,\, t-\Delta)}, e^{(1)}_{(t-\Delta-\nu+1,\, t-\Delta)}) \tag{3.63}$$

$$= Pr(\underline{Z}_{(1,\, t-\Delta)}, e^{(1)}_{(t-\Delta-\nu+1,\, t-\Delta)})$$

since $e^{(1)}_{t-\Delta}$ is contained in $e^{(1)}_{(t-\Delta-\nu+1,\, t-\Delta)}$.

Now

$$g_\Delta = Pr(e^{(1)}_{t-\Delta}, Z_{(1,\, t-\Delta+\Delta)}, e^{(1)}_{(t-\Delta-\nu+1+\Delta,\, t-\Delta+\Delta)})$$

$$= Pr(e^{(1)}_{t-\Delta}, Z_{(1,\, t)}, e^{(1)}_{(t-\nu+1,\, t)}) \tag{3.64}$$

is exactly the term needed in the numerator summation ( 3.54 ) to decide on $\hat{e}^{(1)}_{t-\Delta}$.

Now following the modification of Fritchman and Mixsell [20] to Lee's algorithm to use the minimum delay one can define

$$Q_i \triangleq g_{i+\nu} = Pr(e^{(1)}_{t-\Delta}, Z_{(1,\, t-\Delta+i+\nu)}, e^{(1)}_{(t-\Delta+i+1,\, t-\Delta+i+\nu)})$$

where
$$\tag{3.65}$$

$$Q_0 = g_\nu = Pr(e^{(1)}_{t-\Delta}, Z_{(1,\, t-\Delta+\nu)}, e^{(1)}_{(t-\Delta+1,\, t-\Delta+\nu)})$$

$$= Pr(Z_{1,\, t-\Delta+\nu}, e^{(1)}_{(t-\Delta,\, t-\Delta+\nu)}) \tag{3.66}$$

and

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$$Q_{\Delta-\nu} = g_\Delta = Pr(e_{t-\Delta}, Z_{(1,t)}, e_{(t-\nu+1,t)}) \tag{3.67}$$

which is exactly the probability needed in ( 3.53 ) to decide on $e_{t-\Delta}^{(1)}$. That is the numerator ( 3.45 ) equals

$$Pr(e_{t-\Delta}^{(1)}=i, Z_{(1,t)}) = \sum_{e_{t-\nu+1}^{(1)}} \cdots \sum_{e_t^{(1)}} Pr(e_{t-\Delta}^{(1)}=i, Z_{(1,t)}, e_{(t-\nu+1,t)}^{(1)}) \tag{3.68}$$

which is expressible as

$$Pr(e_{t-\Delta}^{(1)}=j, Z_{(1,t)}) = \sum_{e_{t-\nu+1}^{(1)}} \cdots \sum_{e_t^{(1)}} Q_{\Delta-\nu} \tag{3.69}$$

for $j = 0, 1$, in this binary case.

<u>Recursion for $Q_i$</u>

Given $Q_0$ for $e_{t-\Delta}^{(1)} = j$, $j=0, 1$, it is possible to recursively compute $Q_1, Q_2, \ldots, Q_{\Delta-\nu}$ as shown below. (In the following the superscript of $e_t^{(1)}$ will be dropped to ease notation, i.e. $e_t^{(1)}=e_t$.)

$$Q_i = \sum_{e_{t-\Delta+i}} P(e_{t-\Delta}=j, Z_{(1, t-\Delta+i+\nu)}, e_{(t-\Delta+i, t-\Delta+i+\nu)})$$

$$= \sum_{e_{t-\Delta+i}} P(Z_{t-\Delta+i+\nu}/e_{t-\Delta}=j, Z_{(1, t-\Delta+i+\nu-1)}, e_{t-\Delta+i, t-\Delta+i+\nu})$$

$$\cdot P(e_{t-\Delta}=j, Z_{(1, t-\Delta+i+\nu-1)}, e_{(t-\Delta+i, t-\Delta+i+\nu)})$$

$$= \sum_{e_{t-\Delta+i}} P(Z_{t-\Delta+i+\nu}/e_{(t-\Delta+i, t-\Delta+i+\nu)})P(e_{t-\Delta}=j, Z_{(1, t-\Delta+i+\nu-1)},$$

$$e_{(t-\Delta+i, t-\Delta+i+\nu)})$$

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$$= \sum_{e_{t-\Delta+i}} P(e_{t-\Delta+i+\nu}) \overbrace{P(e_{t-\Delta}=j, Z_{(1, t-\Delta+i+\nu-1)}, e_{(t-\Delta+i, t-\Delta+i+\nu-1)})}^{Q_{i-1}}$$

$$\cdot P(Z_{t-\Delta+i+\nu}/e_{(t-\Delta+i, t-\Delta+i+\nu)}) \cdot \qquad (3.70)$$

Therefore,

$$\boxed{Q_i = \sum_{e_{t-\Delta+i}} P(e_{t-\Delta+i+\nu}) P(Z_{t-\Delta+i+\nu}/e_{(t-\Delta+i, t-\Delta+i+\nu)}) Q_{i-1}} \qquad (3.71)$$

Note that since both numerators (one for $e_{t-\Delta}=0$ and one for $e_{t-\Delta}=1$) must be computed, at each step in the recursion there are two $Q_i$'s.

Decision

Once the two parallel recursions are complete (yielding $Q_{\Delta-\nu}$ for $e_{t-\Delta} = 0$ and $e_{t-\Delta} = 1$) one can then decide on $\hat{e}_{t-\Delta}$ by comparing the numerator for $e_{t-\Delta} = 0$,

$$P(e_{t-\Delta}=0, Z_{(1, t)}) = \sum_{e_{t-\nu+1}} \cdots \sum_{e_t} P(e_{t-\Delta}=0, Z_{(1, t)}, e_{(t-\nu+1, t)})$$

$$= \sum_{e_{t-\nu+1}} \cdots \sum_{e_t} Q_{\Delta-\nu} \text{ (with } e_{t-\Delta}=0)$$

$$(3.72)$$

with the numerator for $e_{t-\Delta}=1$,

$$P(e_{t-\Delta}=1, Z_{(1, t)}) = \sum_{e_{t-\nu+1}} \cdots \sum_{e_t} P(e_{t-\Delta}=1, Z_{(1, t)}, e_{(t-\nu+1, t)})$$

$$= \sum_{e_{t-\nu+1}} \cdots \sum_{e_t} Q_{\Delta-\nu} \text{ (with } e_{t-\Delta}=1) \qquad (3.73)$$

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<u>Determination of $Q_0$</u>

The two starting probabilities $Q_0(e_{t-\Delta}=0)$ and $Q_0(e_{t-\Delta}=1)$ are needed to initiate the above Q-recursion. These may be obtained as follows:

Define

$$q(e_{(t-\Delta,\, t-\Delta+\nu)}) = Q_0 = P(Z_{(1,\, t-\Delta+\nu)},\, e_{(t-\Delta,\, t-\Delta+\nu)}) \quad (3.74)$$

and in general

$$q(e_{(t-\nu+i,\, t+i)}) = P(Z_{(1,\, t+i)},\, e_{(t-\nu+i,\, t+i)}) . \quad (3.75)$$

This q function provides the necessary starting probabilities $Q_0$ in a manner similar to Lee [11] in which a denominator recursion was used to provide the start for each numerator recursion.

By using conditioning and averaging one can break up the joint probability for general i and write

$$q(e_{(t-\nu+i,\, t+i)}) = P(e_{t+i})P(Z_{t+i}/e_{(t-\nu+i,\, t+i)}) \sum_{e_{t-\nu+i-1}} q(e_{(t-\nu+i-1,\, t+i-1)}) \quad (3.76)$$

Below is an illustration of the computation of a specific case, namely $i = \nu-\Delta$, of the determination of $Q_0$.

$$Q_0 = q(e_{(t-\Delta,\, t-\Delta+\nu)}) = P(e_{(t-\Delta,\, t-\Delta+\nu)},\, Z_{(1,\, t-\Delta+\nu)})$$

$$= P(Z_{t-\Delta+\nu}/Z_{(1,\, t-\Delta+\nu-1)},\, e_{(t-\Delta,\, t-\Delta+\nu)})P(Z_{(1,\, t-\Delta+\nu-1)},\, e_{(t-\Delta,\, t-\Delta+\nu)})$$

$$= P(Z_{t-\Delta+\nu}/e_{(t-\Delta,\, t-\Delta+\nu)})P(Z_{(1,\, t-\Delta+\nu-1)},\, e_{(t-\Delta,\, t-\Delta+\nu)})$$

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$$= P(Z_{t-\Delta+\nu}/e_{(t-\Delta,\,t-\Delta+\nu)})P(e_{t-\Delta+\nu})P(Z_{(1,\,t-\Delta+\nu-1)},\,e_{(t-\Delta,\,t-\Delta+\nu-1)}/e_{t-\Delta+\nu})$$

$$= P(e_{t-\Delta+\nu})P(Z_{t-\Delta+\nu}/e_{(t-\Delta,\,t-\Delta+\nu)})P(Z_{(1,\,t-\Delta+\nu-1)},\,e_{(t-\Delta,\,t-\Delta+\nu-1)})$$

$$= P(e_{t-\Delta+\nu})P(Z_{t-\Delta+\nu}/e_{(t-\Delta,\,t-\Delta+\nu)})\sum_{e_{t-\Delta-1}}\underbrace{P(Z_{(1,\,t-\Delta+\nu-1)},\,e_{(t-\Delta-1,\,t-\Delta+\nu-1)})}_{q(e_{(t-\Delta-1,\,t-\Delta+\nu-1)})}$$

$$(3.77)$$

Thus it is feasible to obtain $Q_0$ from the previous value of $q(e_{(t-\Delta-1,\,t-\Delta+\nu-1)})$. One should note that $P(e_{t-\Delta+\nu})$ in the above is known for the binary symmetric channel. The second probability, $P(Z_{t-\Delta+\nu}/e_{(t-\Delta,\,t-\Delta+\nu)})$ must be averaged over the "don't care" error(s) namely,

$$P(Z_{t-\Delta+\nu}/e_{(t-\Delta,\,t-\Delta+\nu)}) = \sum_{\delta_{t-\Delta+\nu}} P(Z_{t-\Delta+\nu}/e_{(t-\Delta,\,t-\Delta+\nu)},\,\delta_{t-\Delta+\nu})$$

$$\cdot P(\delta_{t-\Delta+\nu}) \qquad (3.78)$$

where $\delta_{t-\Delta+\nu}$ represents those error bits which affect the output of the syndrome former at time $t-\Delta+\nu$ but do not appear in the syndrome former physical state (e.g. $e_{t-\Delta+\nu}^{(2)}$ in the rate $\frac{1}{2}$ example). This type of computation of the syndrome former output probabilities was to be expected since this analysis is done for a _multiple_ input syndrome former. Some inputs (e.g. $e_t^{(1)}$) needed for decoding must be estimated while other inputs (e.g. $e_t^{(2)}$) are nuisance inputs which must be averaged out.

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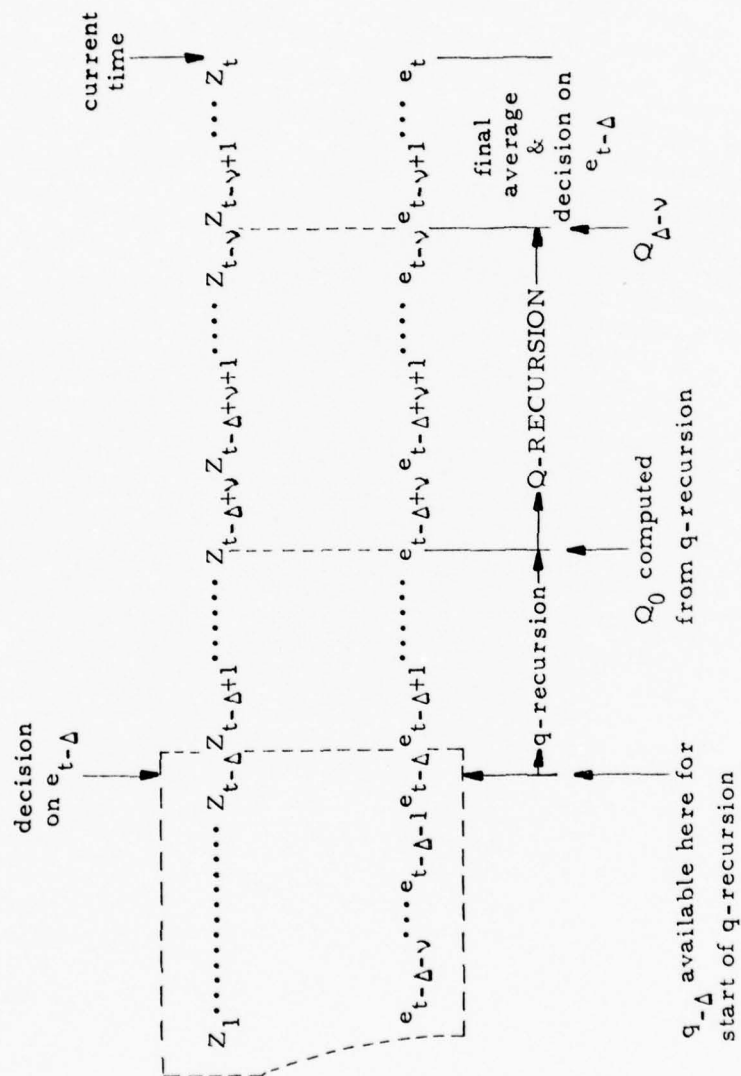
Figure 3.4. Optimal Real Time MAP Error Bit Recursion Relationships

does not have to be adjusted for nonequiprobable $e_t^{(1)}$ inputs. Once

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It is best to note that $Q_0 = q(e_{(t-\Delta, t-\Delta+\nu)})$ is actually a <u>set</u> of $2^{\nu+1}$ numbers representing the joint probabilities $P(Z_{(1, t-\Delta+\nu)},$ $e_{(t-\Delta, t-\Delta+\nu)})$. One can consider each of these numbers to be labeled by the associated binary valued vectors $(e_{t-\Delta}, e_{t-\Delta+1}, \dots, e_{t-\Delta+\nu})$. Thus the initial $Q_0$ starting probabilities for the $Q$ recursion to $Q_{\Delta-\nu}$ is a substantial set of $2^{\nu+1}$ numbers labeled by $\nu+1$ binary error values.

However in the algorithm to follow, one never has to process more than $2^{\nu+1}$ numbers since in the $Q$ recursion the contribution of a new error value is included while that of an old error value is averaged out. Thus the necessary storage for $2^{\nu+1}$ numbers is maintained throughout the recursion to $Q_{\Delta-\nu}$ ($\Delta-\nu$ steps). The final $\nu$ averaging steps following the $Q$ recursion reduce the $2^{\nu+1}$ numbers down to two upon which the decision on $e_{t-\Delta}$ is made. One does, however, need to still retain the $2^{\nu+1}$ values for $Q_0$ until subsequent steps decide $\hat{e}_{t-\Delta}$ since these values are needed to compute the new set of $Q_0$ when t is increased by one to estimate $\hat{e}_{t-\Delta+1}$ in the next pass. Thus the maximum storage required is the sum $2^{\nu+1} + 2^{\nu+1} = 2^{\nu+2}$.

## Algorithm for Real Time MAP Error Bit Estimate

Below is given the optimal real time MAP error bit estimation algorithm assuming a delay of $\Delta$. The statements which are boxed represent the actual algorithm with the remainder being additional

clarifications.

---

Step 0: Set $q(e_{(1,0)}) = 1$ for $t = 0$.

For $t = 1, 2, \ldots, \nu+1$ calculate

$$q(e_{(1,t)}) = Pr(e_t)Pr(Z_t/e_{(1,t)})q(e_{(1,t-1)})$$

---

Note this is the case of Eq. ( 3.76 ) with i=0 and $t \le \nu+1$. Thus no summation is required until t is $> \nu+1$. That is

$$q(e_{(t-\nu+i, t+i)}) = Pr(e_{t+i})Pr(Z_{t+i}/e_{(t-\nu+i, t+i)}) \sum_{e_{t-\nu+i-1}} q(e_{(t-\nu+i-1, t+i-1)})$$

With i=0 one has

$$q(e_{(t-\nu, t)}) = Pr(e_t)Pr(Z_t/e_{(t-\nu, t)}) \sum_{e_{t-\nu-1}} q(e_{(t-\nu-1, t-1)}) \tag{3.79}$$

and for $t \le \nu+1$, $q(e_{(t-\nu, t)}) = q(e_{(1,t)})$. Thus

$$q(e_{(1,t)}) = Pr(e_t)Pr(Z_t/e_{(1,t)})q(e_{(1,t-1)}) \text{ for } t \le \nu+1 .$$

Considering the above, the values of the initial start up q terms can be tabulated as below:

i=0, t=0: $q(e_{(1,0)}) = 1$ (one term) .

i=0, t=1: $q(e_{(1,1)}) = P(e_1)P(Z_1/e_{(1,1)})q(e_{(1,0)})$ (two terms)

this yields:

$$q(e_{(1,1)}=0) = P(e_1=0)P(Z_1/e_1=0)(1) = Pr(e_1=0, Z_1)$$

$$q(e_{(1,1)}=1) = P(e_1=1)P(Z_1/e_1=1)(1) = Pr(e_1=1, Z_1).$$

$i=0, t=2:$ $\quad q(e_{(1,2)}) = P(e_2)P(Z_2/e_{(1,2)})q(e_{(1,1)})$ (four terms)

this yields:

$$q(e_{(1,2)}=00) = P(e_2=0)P(Z_2/e_{(1,2)}=00)q(e_{(1,1)}=0)$$

$$q(e_{(1,2)}=01) = P(e_2=1)P(Z_2/e_{(1,2)}=01)q(e_{(1,1)}=0)$$

$$q(e_{(1,2)}=10) = P(e_2=0)P(Z_2/e_{(1,2)}=10)q(e_{(1,1)}=1)$$

$$q(e_{(1,2)}=11) = P(e_2=1)P(Z_2/e_{(1,2)}=11)q(e_{(1,1)}=1).$$

$$\vdots \qquad\qquad \vdots$$

$i=0, t=\nu+1:$ $\quad q(e_{(1,\nu+1)}) = P(e_{\nu+1})P(Z_{\nu+1}/e_{(1,\nu+1)})q(e_{(1,\nu)})$ $(2^{\nu+1}$ terms)

$$(3.80)$$

The above computations form a tree that is $\nu+1$ branches long having $2^{\nu+1}$ terminal nodes. This "initial error tree" is sketched in Figure 3.5.

> Step 1: Set $P(e_{t-\nu}, Z_{(1,t)}, e_{(t-\nu+1,t)}) = q(e_{(t-\nu,t)})$
>
> If $\Delta \leq \nu$ go to Step 4 (see Appendix C).

For illustration, consider the vase of $t = \nu+1$ which is the first time Step 1 is followed. One thus has,

$$\underbrace{P(e_1, Z_{(1,\nu+1)}, e_{(2,\nu+1)})}_{2^{\nu+1} \text{ terms}} = \underbrace{q(e_{(1,\nu+1)})}_{= 2^{\nu+1} \text{ terms}} = P(Z_{(1,\nu+1)}, e_{(1,\nu+1)})$$

$$(3.81)$$

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$2^{\nu+1}$ distinct
values of
$q(e_{(1, \nu+1)})$

$q(e_{(1, \nu+1)})$

$e_{\nu+1}$

$t=\nu+1$

$q(e_{(1, 3)})$

$q(e_{(1, 2)})$

$q(e_{(1, 1)})$

$q(e_{(1, 0)})$

$e_3$

$t = 3$

$e_2$

$t = 2$

$e_1$

$t = 1$
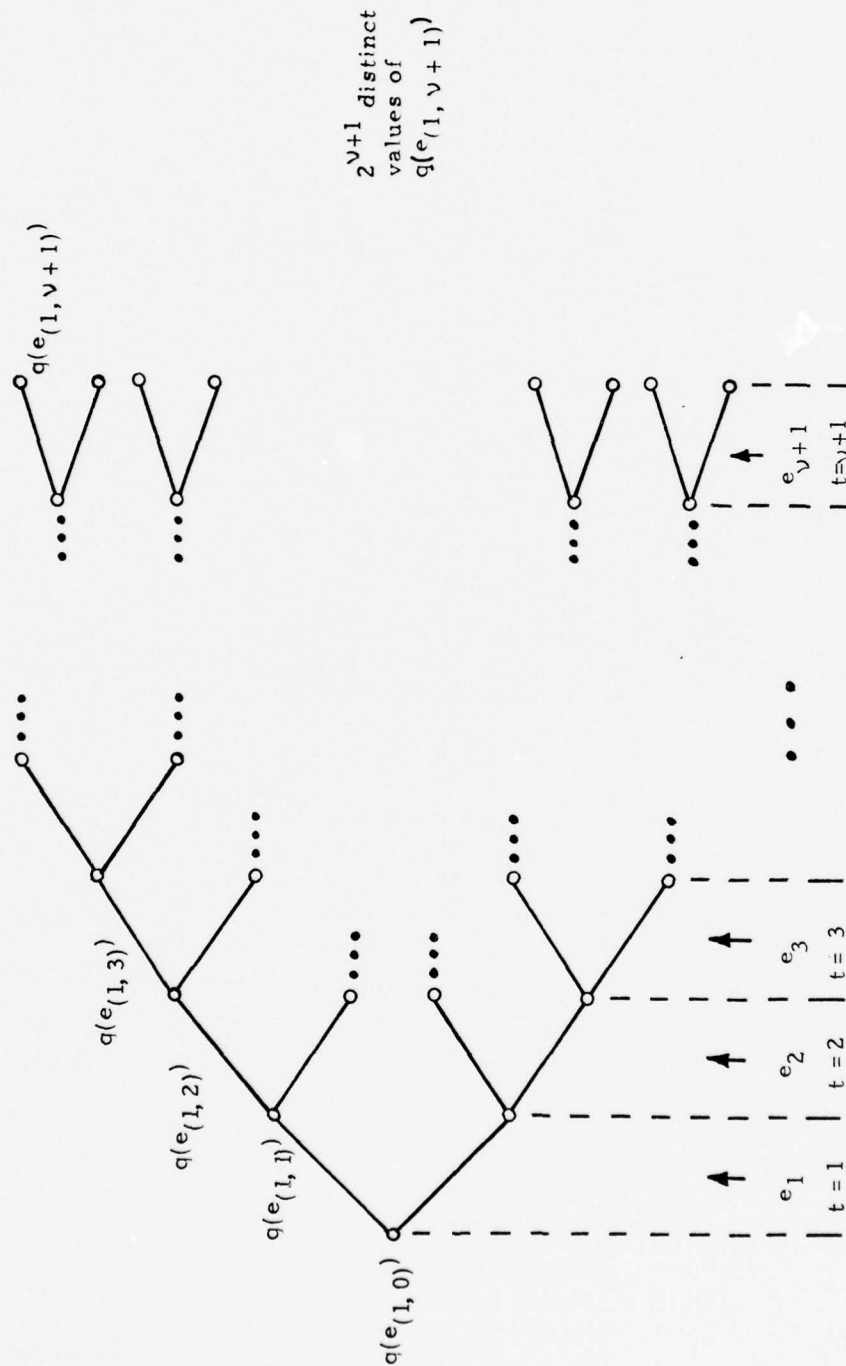
Figure 3.5 Optimal Real Time MAP Error Bit Estimate Initial Error Tree

Step 2: (The Q-recursion for $\Delta > \nu$ case.) For $i=1, 2, \ldots, \Delta-\nu$

calculate

$$P(e_{t-\nu}, Z_{(1,t+i)}, e_{(t-\nu+1+i,t+i)})$$

$$= \sum_{e_{t-\nu+i}} P(e_{t+i})P(Z_{t+i}/e_{(t-\nu+i,t+i)})$$

$$\cdot P(e_{t-\nu}, Z_{(1,t+i-1)}, e_{(t-\nu+i,t+i-1)})$$

Step 2 basically increments by the new error bit and averages over an old error bit. Specifically for $t = \nu+1$ this becomes:

$i=1, Q_i$:

$$P(e_1, Z_{(1,\nu+2)}, e_{(3,\nu+2)})$$

$$= \sum_{e_2} P(e_{\nu+2})P(Z_{\nu+2}/e_{(2,\nu+2)})P(e_1, Z_{(1,\nu+1)}, e_{(2,\nu+1)})$$

$$= \sum_{e_2} P(e_{\nu+2})P(Z_{\nu+2}/e_{(2,\nu+2)})\underbrace{q(e_{(1,\nu+1)})}_{2^{\nu+1} \text{ of these terms}} \qquad (3.82)$$

It is instructive to consider just what the above $Q_1$ calculation actually involves. Recall that $q(e_{(1,\nu+1)})$ is a set of $2^{\nu+1}$ numbers (probabilities) each labeled by values of $(e_1, e_2, \ldots, e_{\nu+1})$ as indicated in the initial error tree of Figure 3.5 . The step 2 for $Q_1$ requires averaging over $e_2$. Therefore one sets $e_{\nu+2}=0$. Next one

$$P(\underline{Z}_t/e_{t-\Delta}, \underline{Z}_{(1,t-1)}, e_{(t-\nu,t)}) = P(\underline{Z}_t/e_{(t-\nu,t)}). \qquad (3.55)$$

takes the pair of numbers $q(e_{(1, e_2=0, 3, \ldots, \nu+1)})$ and

$q(e_{(1, e_2=1, 3, \ldots, \nu+1)})$ and multiplies each by $P(e_{\nu+2}=0) \cdot$

$P(Z_{\nu+2}/e_{(2=0, 3, \ldots, \nu+2=0)})$, and $P(e_{\nu+2}=0)P(Z_{\nu+2}/e_{(2=1, 3, \ldots, \nu+2=0)})$

respectively and adds.

When all pairs of numbers $q(e_{(1, \ldots, \nu+1)})$ have been so

processed one has $2^\nu$ new probabilities corresponding to $e_{\nu+2}=0$.

Now the above process is repeated with $e_{\nu+2}=1$ to yield $2^\nu$ new

probabilities incorporating the influence of $e_{\nu+2} = 1$.

Thus at the $Q_1$ step there must be stored $2^\nu+2^\nu = 2^{\nu+1}$ numbers

to be used to compute $Q_2$. These are of the form: $P(e_1, Z_{(1, \nu+2)}, $

$e_{(3, \nu+2)})$. By "incrementing" by $e_{\nu+2}$ and averaging over $e_2$, the

required storage is maintained at $2^{\nu+1}$ as it can be also for the rest

of the Q recursion to $Q_{\Delta-\nu}$. However the old set of $q(e_{(1, \nu+1)})$ must

be kept in storage for use in Step 5 so the actual storage necessary

is $2^{\nu+2}$.

Continuing the Q-recursion gives at $i = \Delta-\nu, Q_{\Delta-\nu}$:

$$P(e_1, Z_{(1, 1+\Delta)}, e_{(2+\Delta-\nu, 1+\Delta)})$$

$$= \sum_{e_{1+\Delta-\nu}} P(e_{1+\Delta})P(Z_{1+\Delta}/e_{(1+\Delta-\nu, 1+\Delta)}) \underbrace{P(e_1, Z_{(1, \Delta)}, e_{(1+\Delta-\nu, \Delta)})}_{Q_{\Delta-\nu-1}}$$

$$(3.83)$$

Thus following the $Q_{\Delta-\nu}$ step of the Q-recursion, the decoder

has stored $2^{\nu+1}$ probabilities labeled by the binary vectors

$(e_1, e_{\Delta-\nu+2}, \ldots, e_{1+\Delta})$. These probabilities are

$$p(e_1, Z_{(1, 1+\Delta)}, e_{(\Delta-\nu+2, \Delta+1)}).$$

Now Step 3 is entered to average out $e_{\Delta-\nu+2}$ through $e_{\Delta+1}$.

---

Step 3: (Final average and decision for $\Delta > \nu$ case.) For $i = 0, 1$

calculate

$$\Lambda_i = \sum_{e_{t+\Delta-2\nu+1}} \cdots \sum_{e_{t+\Delta-\nu}} P(e_{t-\nu}=i, Z_{(1, t+\Delta-\nu)}, e_{(t+\Delta-2\nu+1, t+\Delta-\nu)})$$

If $\Lambda_0 > \Lambda_1$ decide $\hat{e}_{t-\nu} = 0$, otherwise $\hat{e}_{t-\nu} = 1$. Go to Step 5 to

"increment" q.

---

For estimating $\hat{e}_1$ with $t = \nu+1$ these quantities are

$$\Lambda_0 = \sum_{e_{\Delta-\nu+2}} \cdots \sum_{e_{\Delta+1}} P(e_1=0, Z_{(1, \Delta+1)}, e_{(\Delta-\nu+2, \Delta+1)})$$

(3.84)

$$\Lambda_1 = \sum_{e_{\Delta-\nu+2}} \cdots \sum_{e_{\Delta+1}} P(e_1=1, Z_{(1, \Delta+1)}, e_{(\Delta-\nu+2, \Delta+1)})$$

Again it is instructive to state just what is being done here.

For $\Lambda_0$ select those numbers of the set $Q_{\Delta-\nu} = \{P(e_1, Z_{(1, \Delta+1)},$ $e_{(\Delta-\nu+2, \Delta+1)})\}$ labeled by $e_1 = 0$. Call this subset $Q_{\Delta-\nu}(e_1=0)$. It contains $2^\nu$ numbers. At the first summation over $e_{\Delta-\nu+2}$ select those numbers in $Q_{\Delta-\nu}(e_1=0)$ whose labels match in all but the

$e_{\Delta-\nu+2}$ position

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$$P(e_1 = 0, Z_{(1, \Delta+1)}, e_{(\Delta-\nu+2=0, \ldots, \Delta+1)}),$$

$$P(e_1 = 0, Z_{(1, \Delta+1)}, e_{(\Delta-\nu+2=1, \ldots, \Delta+1)})$$

and add such pairs equally weighted to obtain $2^{\nu-1}$ elements in the set of new probabilities. Repeat this process a total of $\nu$ times resulting in

$$\Lambda_0 = P(e_1 = 0, Z_{(1, \Delta+1)})$$

$$\Lambda_1 = P(e_1 = 1, Z_{(1, \Delta+1)})$$

If $\Lambda_0 > \Lambda_1$ decide $\hat{e}_1 = 0$, otherwise $\hat{e}_1 = 1$.

---

Step 5: (Increment q.) Increase t by 1 and calculate

$$q(e_{(t-\nu, t)}) = P(e_t) P(Z_t / e_{(t-\nu, t)}) \sum_{e_{t-\nu-1}} q(e_{(t-\nu-1, t-1)})$$

Go to Step 1.

---

Here for example $t = \nu+1$ would increase to $t = \nu+2$ to give

$$q(e_{(2, \nu+2)}) = P(e_{\nu+2}) P(Z_{\nu+2} / e_{(2, \nu+2)}) \sum_{e_1} q(e_{(1, \nu+1)}) \qquad (3.85)$$

Here for each of the $2^{\nu+1}$ numbers $q(e_{(1, \nu+1)})$ in the set from the initial error tree, find the pair $q(e_1 = 0, e_2, \ldots, e_{\nu+1})$, $q(e_1 = 1, e_2, \ldots, e_{\nu+1})$ whose labels differ only in the $e_1$ position and add. This yields $2^\nu$ new numbers. Next produce $2^{\nu+1}$ numbers by

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multiplying <u>each</u> by $P(e_{\nu+2}=0)P(Z_{\nu+2}/e_{(2,\ldots,\nu+1,\nu+2=0)})$ and by

$P(e_{\nu+2}=1)P(Z_{\nu+2}/e_{(2,\ldots,\nu+1,\nu+2=1)})$. This gives a new effective

error tree of $2^{\nu+1}$ values of $q(e_{(2,\nu+2)})$ for use in deciding $\hat{e}_2$ during

the next pass. Now go to Step 1 to process for $\hat{e}_2$, etc.

Thus the complete algorithm for estimating the error bit $e_{t-\Delta}$

at time t has been stated. This algorithm represents the optimal

approach since the complete history of syndromes, namely $Z_{(1,t)}$

has been used to make the MAP error bit estimate with a finite

delay $\Delta$. This approach should provide performance the same as

that of a MAP estimation of information bits. However the actual

implementation is complicated by the non-equilikely nature of the

error bits. This requires use of actual probabilities or logarithms

of probabilities in the error bit case as opposed to the simpler

Hamming distance metric useable in a Viterbi decoder for inform-

ation bit sequences. With such a drawback due to the associated

computation overhead, one is tempted to investigate possible

suboptimal alternatives to the above using a finite length syndrome

vector such as $Z_{(t-\Delta,t)}$. This will be done in Chapter 4.

The algorithm given above is complete for the case $\Delta > \nu$. For

the case $\Delta \le \nu$ the reader is referred to Appendix C which gives the

missing Step 4 needed if $\Delta \le \nu$.

### 3.4 Real Time Viterbi MAP Error Sequence Decoding

Conventional Viterbi data sequence decoding of convolutional codes has been shown by Viterbi [14] and Heller and Jacobs [17] to be a very attractive approach for the equilikely data situation. In such a case, the Viterbi algorithm has been shown by Viterbi [14] to choose $\hat{I}_{(1, N)}$ to be the information (data) sequence which maximizes the conditional probability $Pr(I_{(1, N)}/\underline{R}_{(1, N+\nu)})$. When all input data sequences $I_{(1, N)}$ are equilikely, significant simplification of the metric computations is possible. Indeed MAP and Maximum Likelihood estimators have the same form.

As another example of how conventional decoding algorithms adapt to error decoding, the conventional Viterbi approach will now be extended to error sequence estimation. This will require incorporation of the non-equilikely nature of the error inputs to the syndrome former and will assume a decoding delay requirement of $\Delta$. Recall that for a BSC, higher weight error sequences are much less probable than low weight error sequences. Another complication is that, unlike the information sequence case, not all error bits enter into the syndrome former physical state. Some (e.g. $e_t^{(2)}$ in the rate $\frac{1}{2}$ systematic case) modify the syndrome former output $Z_t$ but never actually residue in a syndrome former storage element. Making these two adjustments, the Real-Time Viterbi Decoding Algorithm of Lee [11] is rederived below for error sequences.

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For a syndrome decoder with delay $\Delta$, the <u>Viterbi Decoding Rule</u>

<u>for Error Sequences</u> would be:

Choose $\hat{e}_t^{(1)}$ as the first bit of the syndrome former state

$S_{t+1}$ in the state sequence $S_{(1, t+\Delta+1)}$ which maximizes

the conditional probability

$$\Pr(S_{(1, t+\Delta+1)} / Z_{(1, t+\Delta)}) = \frac{\Pr(S_{(1, t+\Delta+1)}, Z_{(1, t+\Delta)})}{\Pr(Z_{(1, t+\Delta)})} \tag{3.86}$$

As in previous work the denominator is the same for all

candidate state sequences. Thus the necessary maximization

can be done on the joint probability in the numerator.

The syndrome former decoder for the K=5 code with finite

delay is shown again in part in Figure 3.6   to clarify state and

error bit relationships.

<u>Recursion for Joint Probability</u>

The numerator joint probability may be converted to a recursive

form as follows:

$$\Pr(S_{(1, t+1)}, Z_{(1, t)}) = \Pr(S_{(1, t)}, S_{t+1}, Z_{(1, t-1)}, Z_t)$$

$$= \Pr(S_{(1, t)}, Z_{(1, t-1)}) \Pr(S_{t+1}, Z_t / S_{(1, t)} Z_{(1, t-1)})$$

$$= \Pr(S_{(1, t)}, Z_{(1, t-1)}) \Pr(S_{t+1}, Z_t / S_t)$$

$$\underbrace{\Pr(S_{(1, t+1)}, Z_{(1, t)})}_{\text{New Joint Probability}} = \underbrace{\Pr(S_{(1, t)}, Z_{(1, t-1)})}_{\substack{\text{Old Joint Prob-}\\\text{ability}}} \underbrace{\Pr(S_{t+1} / S_t)}_{\substack{\text{State Trans-}\\\text{ition Prob-}\\\text{ability}}} \underbrace{\Pr(Z_t / S_t, S_{t+1})}_{\substack{\text{Syndrome}\\\text{Former}\\\text{Output}\\\text{Probability}}}$$

$$\tag{3.87}$$

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Syndrome Former State = $S_t = [e_{t-1}^{(1)}, e_{t-2}^{(1)}, \ldots, e_{t-4}^{(1)}]$
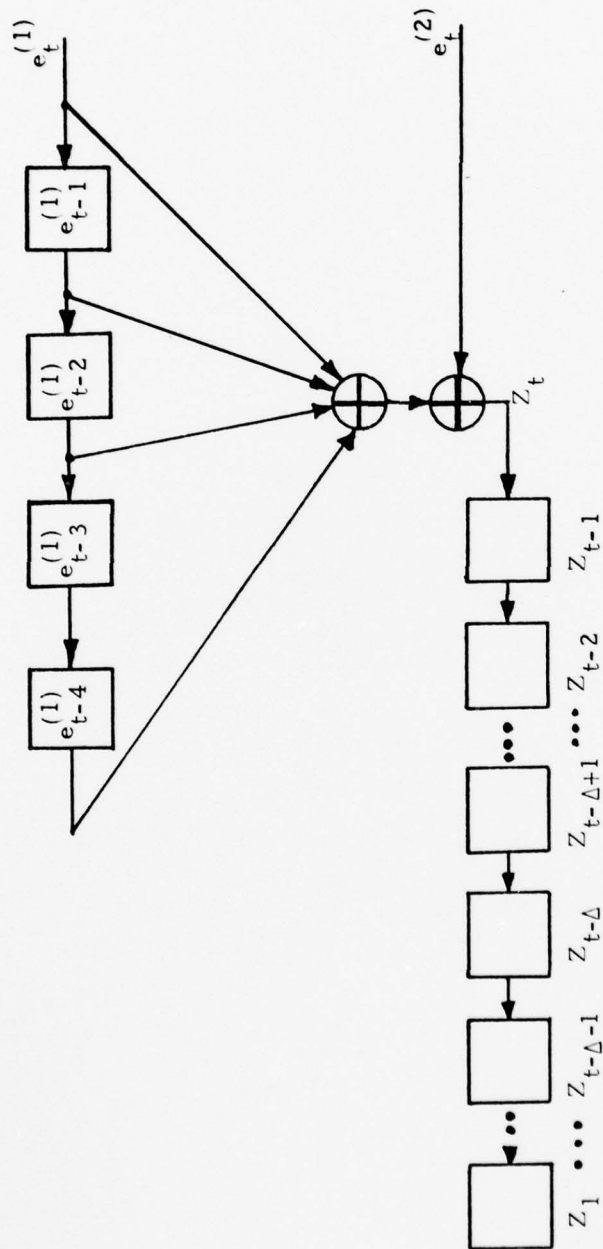
Syndrome Vector = $Z_{(1,t)} = [Z_t, Z_{t-1}, \ldots, Z_{t-\Delta}, \ldots, Z_1]$

Figure 3.6  Syndrome Former and Vector for K=5 (G2=11101) Code

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In the above, the fact that the joint probability of $S_{t+1}$ and $Z_t$, when conditioned on $S_t$, is not a function of either $Z_{(1,t-1)}$ or $S_{(1,t-1)}$ has been applied.

The <u>Syndrome Former Output Probability</u> may be computed by using:

$$Pr(Z_t/S_t, S_{t+1}) = \sum_{\mathscr{E}_t} Pr(Z_t, \mathscr{E}_t/S_t, S_{t+1})$$

$$= \sum_{\mathscr{E}_t} Pr(Z_t/S_t, S_{t+1}, \mathscr{E}_t) Pr(\mathscr{E}_t) \qquad (3.88)$$

where $\mathscr{E}_t$ = (those error bits in $\underline{E}_t$ which affect the syndrome $Z_t$ but which do not appear in the syndrome former physical state (e.g. $e_t^{(2)}$ in K=5 example).)

The <u>Syndrome Former Transition Probability</u> may be computed by using:

$$Pr(S_{t+1}/S_t) = \sum_{E_t^*} Pr(S_{t+1}, E_t^*/S_t)$$

$$= \sum_{E_t^*} Pr(S_{t+1}/E_t^*, S_t) Pr(E_t^*) \qquad (3.89)$$

where $E_t^*$ = (those error bits in $\underline{E}_t$ which actually enter into the syndrome former physical state (e.g. $e_t^{(1)}$ in the K=5 example).

Note that, unlike the conventional Viterbi data decoding case, syndrome former transition probabilities are <u>not</u> limited to 0 or $\frac{1}{2}$.

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Rather they are determined by the channel error probability as represented by $E_t^*$.

## Initial Conditions

If it is assumed that the syndrome former physical state at $t=1$ is $\underline{0}$, then the initial conditions for the above recursion are:

$$\Pr(S_{(1,2)}, Z_{(1,1)}) = \begin{cases} (1)\ \Pr(S_2/\underline{0})\Pr(Z_1/\underline{0}, S_2) \\ \text{if } \underline{0} \in \mathcal{O}(S_2) \text{ and} \\ 0 \text{ if } \underline{0} \notin \mathcal{O}(S_2) \end{cases} \quad (3.90)$$

Here

$$\Pr(S_2/\underline{0}) = \sum_{E_1^*} \Pr(S_2/E_1^*, \underline{0})\Pr(E_1^*), \quad (3.91)$$

$\mathcal{O}(S_i) = $ the allowed predecessor states which could transition to $S_i$,

and

$$\Pr(Z_1/\underline{0}, S_2) = \sum_{\mathcal{S}_t} \Pr(Z_1/\underline{0}, S_2, \mathcal{S}_t)\Pr(\mathcal{S}_t). \quad (3.92)$$

The best sequence $S_{(1,t+1)}$ for maximizing $\Pr(S_{(1,t+1)}, Z_{(1,t)})$ at time $t+1$ with $S_{t+1} = s$ must be the extension of one of the best sequences $S_{(1,t)}$ found at time $t$ with $S_t$ being one of the predecessors of $s$ (i.e. $S_t\ \mathcal{O}(s)$). In this error sequence decoding case each state $s$ may have a number of predecessors. This condition gives rise to the more complex syndrome former state and trellis diagrams as noted in Schalkwijk and Vinck [13]. Thus in a Viterbi Error Sequence

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Algorithm at time $t+1$, each syndrome former state s must have corresponding storage for only the best sequence to state s and the associated joint probability $\Pr(S_{(1, t+1)}, Z_{(1, t)})$.

The above adaptation has considered the entire state sequence $S_{(1, t)}$. For real time (i.e. finite delay = $\Delta$) Viterbi error sequence decoding, only $\Delta$ elements of the best sequence for each state must be stored along with $\Pr(S_{(1, t)}, Z_{(1, t-1)})$.

Adaptation of Lee's Viterbi Algorithm

With the above clarifications, Lee's Real Time Viterbi Decoding Algorithm can be easily modified into the Real Time Viterbi Error Sequence Decoding Algorithm. Following the notation of Lee [11], the necessary metrics and sequences can be defined as: $h(s) = \Pr(S^*_{(1, t+1)}, Z_{(1, t)})$ for all syndrome former states where $S^*_{(1, t+1)}$ is the best path $S_{(1, t+1)}$ with $S_{t+1} = s$ and $B_i(s) = $ first digit of state $S_{t+1-i}$, in the path $S^*_{(1, t+1)}$. Note that $B_i(s) = \hat{e}^{(1)}_{t-i}$ in the example case of a rate $1/n$ systematic code.

Thus the following algorithm can be given.

Real Time Viterbi Error Sequence Decoding Algorithm $(\Delta > \nu)$

Step 0: Set $t=1$. Set $h(s) = \Pr(S^*_{(1, 2)}, Z_{(1, 1)}) = \Pr(s/\underline{0})\Pr(Z_1/\underline{0}, s)$ for all states having $\underline{0}$ as a predecessor. Set $h(s) = 0$ otherwise. Set $B_i(s) = 0$ for $1 \le i \le \Delta$ and for all states s.

Step 1: If $t \le \Delta$ go to Step 2. Otherwise set the error estimate $\hat{e}^{(1)}_{t-\Delta} = B_\Delta(s)$ where s is the state for which $h(s)$ is a maximum.

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<u>Step 2</u>: Increase t by 1. Replace $B_{\Delta-i+1}(s) \leftarrow B_{\Delta-i}(s)$ for $i = 1, 2, \ldots,$

$\Delta-1$ for each state s.

<u>Step 3</u>: For every s make the replacement $h(s) \leftarrow Pr(s/s')Pr(Z_t/s, s') \cdot$

h(s) where s' is the state maximizing the quantity on the right.

Set $B_1(s) = $ 1st digit of s' for each s. Return to Step 1.

It has not been difficult in principle to adapt a finite delay Viterbi

data sequence decoding algorithm to the decoding of error sequences.

In practice, implementation of a Viterbi error sequence decoder

would be more complex than a conventional Viterbi data sequence

decoder. Unlike data sequences, which can be assumed equilikely,

error sequences are very non-equiprobable. Thus the metrics

involved in a Viterbi approach must be actual probabilities (or

logarithms of probabilities) not merely simple functions of Hamming

distances between received and candidate transmitted sequences.

Appendix A, "Viterbi Algorithm for MAP Estimation," has additional

details of the adaptation of the Viterbi Algorithm necessary with

non-equilikely inputs.

## 3.5 Relation of Syndrome Decoding to MAP Data Bit Decoding

In any kind of MAP decoding the goal is to estimate some

quantity A given some other quantity B by finding that value of A

denoted by $\hat{A}$ which maximizes $P(A/B)$.

Since

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$$P(A/B) = \frac{P(A, B)}{P(B)} = \frac{P(B/A)P(A)}{P(B)} \qquad (3.93)$$

one can either

(1) determine a computable form of the expression for $P(A/B)$
   <u>directly</u> and then find the $\hat{A}$ which causes $P(A/B)$ to be
   maximum; or

(2) determine a computable form of the expression $P(B/A)P(A)/P(B)$
   and determine $\hat{A}$ which causes $P(B/A)P(A)/P(B)$ to be maximum.

   With either approach one basic assumption is that the quantities
$A$ and $B$ are <u>related</u> by some, usually probabilistic, relation or
function.

   Suppose we now apply the above to the problem of decoding
convolutional codes. Here the fundamental problem is given a
received vector $\underline{R}$ determine the best estimate of a particular
information bit $I_t$. If we desire to use a MAP estimate then ( 3.93 )
becomes:

$$P(I_t/\underline{R}) = \frac{P(\underline{R}/I_t)P(I_t)}{P(\underline{R})} \qquad (3.94)$$

and one tries to determine that value of $I_t$ (call it $\hat{I}_t$) which causes
<u>either</u> side of ( 3.94 ) to be maximum. For example, this is the
approach taken by McAdam [ 9 ] in deriving the MAP estimate of $I_t$
given $\underline{R}$ by expressing the right hand side of ( 3.94 ) in terms of
the encoder state sequences.

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## Usefulness of Syndromes

Now suppose one wants to investigate the estimation of $I_t$ using a computed syndrome ($\underline{Z}$). One is tempted to write ( 3.93 ) as

$$P(I_t/Z) = \frac{P(\underline{Z}/I_t)P(I_t)}{P(\underline{Z})} \qquad (3.95)$$

and try to determine that value of $I_t$ which maximizes either side of ( 3.95 ). However this would be <u>wrong</u> since the basic assumption that $I_t$ is somehow related to $\underline{Z}$ is not true. Since the Syndrome $\underline{Z}$ is only related to the error bits $\{e_j^{(i)}\}$ and is independent of $I_t$, no matter how intricately one expands ( 3.95 ) even in terms of averaging over received vectors $\underline{R}$, the final result is equivalent to

$$P(I_t/\underline{Z}) = P(I_t) \qquad (3.96)$$

<u>Therefore we cannot estimate $I_t$ given Z only.</u>

Actually trying to estimate $I_t$ given $\underline{Z}$ is the Data Processing Theorem at its worst. By the Data Processing Theorem, we can do no better than estimating $I_t$ based on $P(I_t/R)$ since $\underline{R}$ is our received "raw" data. Any "data processor" (e.g. syndrome former) operating on the "raw data" $\underline{R}$ to yield some output (e.g. $\underline{Z}$) will tend to cause some information loss. In our syndrome case, this loss is <u>all</u> the information regarding $I_t$. Since the errors $(e_t)$ are independent of the data $(I_t)$ and the syndrome former output $\underline{Z}$ has no information regarding $I_t$, $\underline{Z}$ still has all the information regarding the errors $\{e_t^{(i)}\}$.

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Consequently, we can just as well estimate $e_t$ using $\underline{Z}$ by maximizing $P(e_t/\underline{Z})$ as we can estimate $e_t$ using $\underline{R}$ by maximizing $P(e_t/\underline{R})$. If we obtain an estimate $\hat{e}_t$ by either method we must still go back to the raw data $\underline{R}$ to estimate the desired $\hat{I}_t$.

Since finite delay $\Delta$ and a discrete memoryless AWGN channel have been assumed, the error estimate $\hat{e}_{t-\Delta}^{(i)}$ can be subtracted from the appropriate $r_{t-\Delta}^{(i)}$ to yield $\hat{C}_{t-\Delta}^{(i)}$:

$$\hat{C}_{t-\Delta}^{(i)} = \underbrace{r_{t-\Delta}^{(i)}}_{\text{known}} - \underbrace{\hat{e}_{t-\Delta}^{(i)}}_{\text{estimated}} \qquad (3.97)$$

With $\hat{C}_t^{(i)}$ estimated available, one can compute $\hat{I}_t$ deterministically. For the binary systematic rate $1/2$ case sketched in Figure 3.7 , this is especially simple since $\hat{C}_t^{(1)} = \hat{I}_t$.

In the binary estimate case shown

$$I_t = C_t^{(1)} = r_t^{(1)} \oplus e_t^{(1)} . \qquad (3.98)$$

A MAP estimator for $I_t$ would use the rule:

<u>Rule 1:</u>

$$\begin{array}{c} \text{decide} \\ \hat{I}_t = 0 \\ P(I_t = 0/\underline{R}) \overset{\geq}{\underset{<}{}} P(I_t = 1/\underline{R}) \\ \text{decide} \\ \hat{I}_t = 1 \end{array} \qquad (3.99)$$

If this rule is rewritten using the fact that $I_t = r_t^{(1)} \oplus e_t^{(1)}$ and the fact that $r_t^{(1)} \epsilon \underline{R}$, one obtains

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Encoder G

$\underline{C}_t$

$\underline{E}_t$

$\underline{R}_t$

$r_t^{(1)}$

Data Processor (syndrome former)

Delay $\Delta$

$\underline{Z}_t$

$r_{t-\Delta}^{(1)}$

MAP estimate of $e_{t-\Delta}^{(1)}$ given $\underline{Z}$

$P(e_{t-\Delta}^{(1)} / \underline{Z})$

$\hat{e}_{t-\Delta}^{(1)}$

$\hat{I}_{t-\Delta}$

$\hat{c}_t^{(1)} = r_t^{(1)} - \hat{e}_t^{(1)} = c_t^{(1)} + e_t^{(1)} - \hat{e}_t^{(1)} = I_t \oplus e_t^{(1)} \oplus \hat{e}_t^{(1)}$

Thus $\hat{I}_t = r_t^{(1)} \oplus \hat{e}_t^{(1)}$ for systematic codes
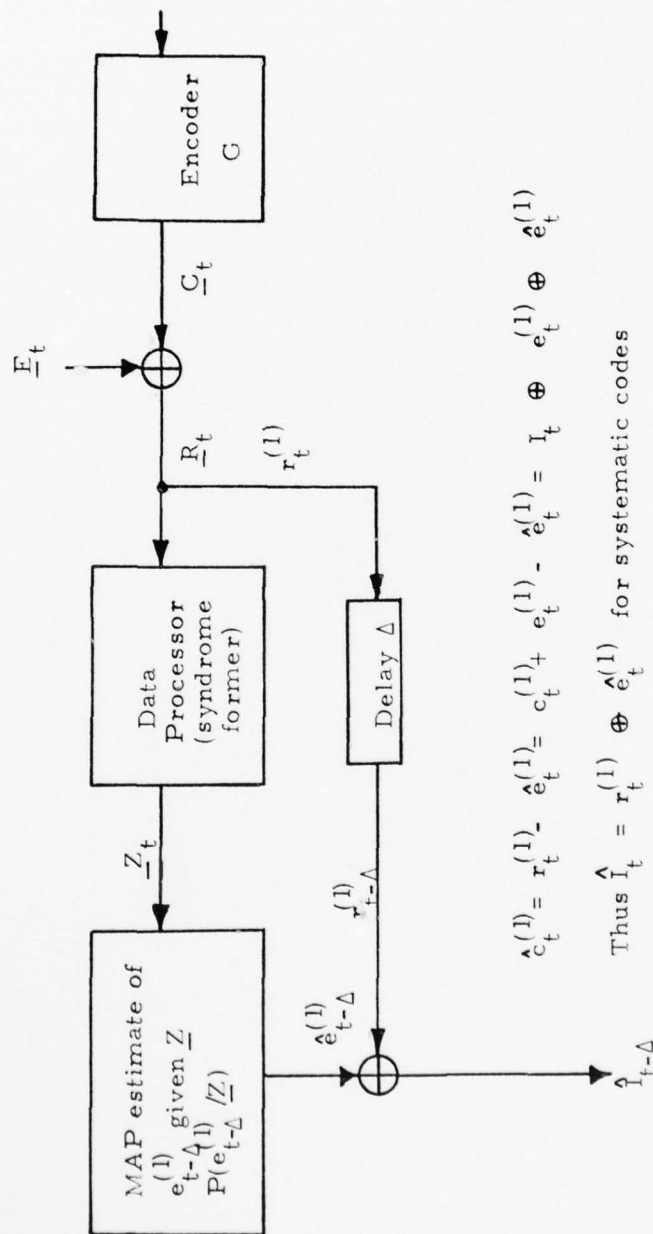
Figure 3.7 Simplified Sketch of Error Decoding of Binary Systematic Rate 1/2 Convolutional Code

1.0

1.1

1.25  1.4  1.6

2.8  2.5
3.2  2.2
3.6
4.0  2.0

1.8

MICROCOPY RESOLUTION TEST CHART
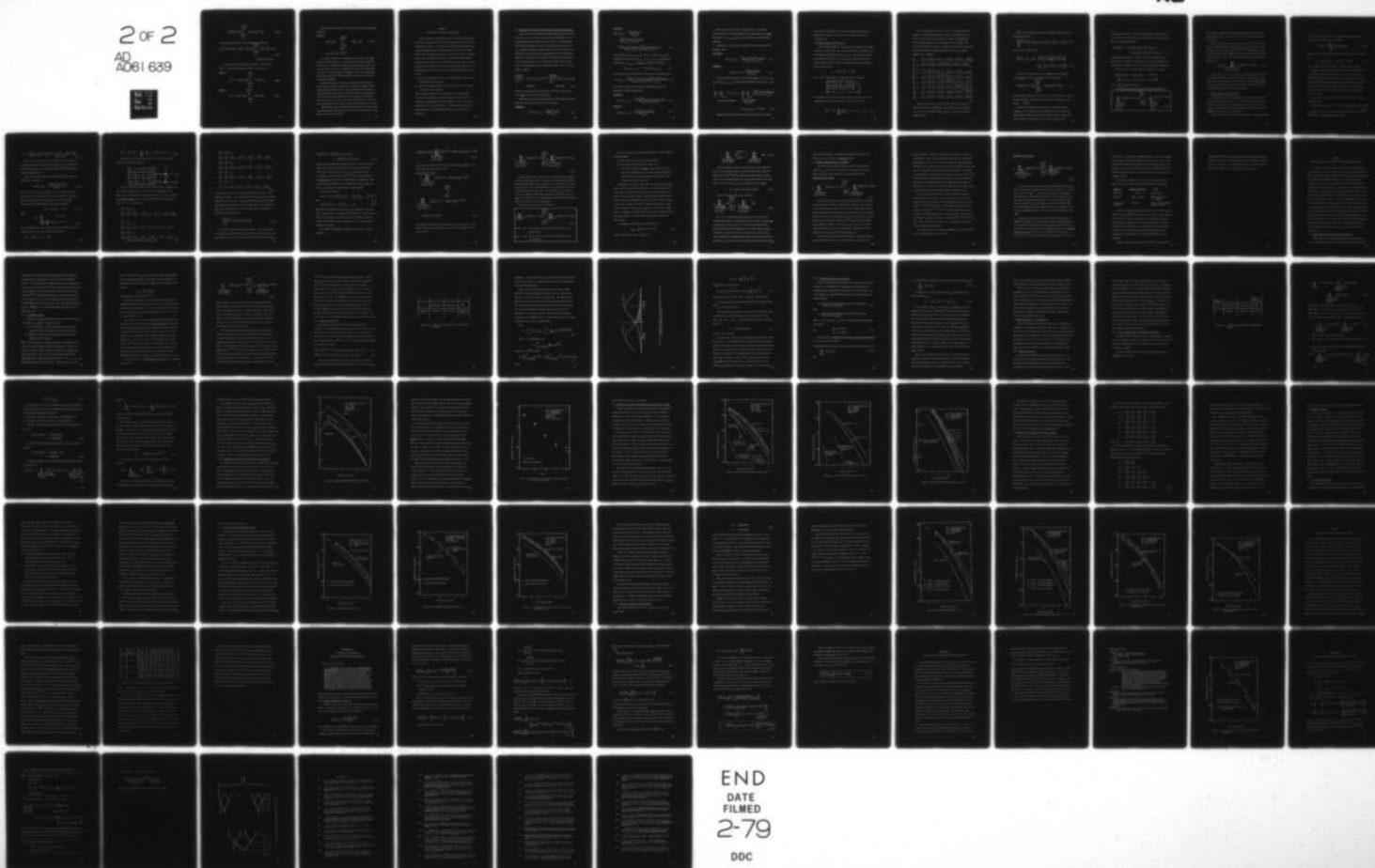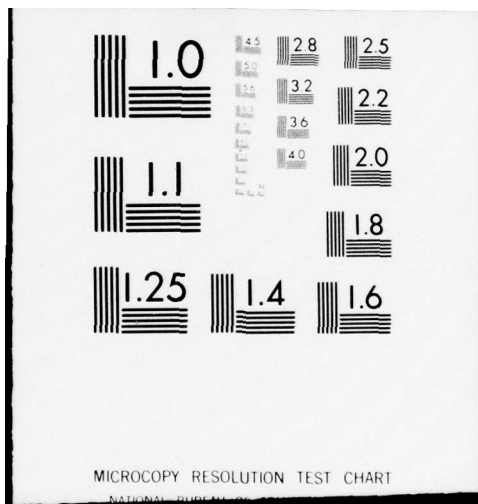
$$\text{P}(r_t^{(1)} \oplus e_t^{(1)} = 0/\underline{R}) \overset{\overset{\overset{\text{decide}}{\hat{I}_t=0}}{\gtrless}}{\underset{\underset{\hat{I}_t=1}{\text{decide}}}{}} \text{P}(r_t^{(1)} \oplus e_t^{(1)} = 1/\underline{R}) \ . \qquad (3.100)$$

The binary (mod 2) sums can be split into the following:

$$\text{P}(r_t^{(1)}=0, e_t^{(1)}=0/\underline{R}) + \text{P}(r_t^{(1)}=1, e_t^{(1)}=1/\underline{R}) \overset{\overset{\overset{\text{decide}}{\hat{I}_t=0}}{\gtrless}}{\underset{\underset{\hat{I}_t=1}{\text{decide}}}{}} \text{P}(r_t^{(1)}=0, e_t^{(1)}=1/\underline{R})$$

$$\qquad (3.101)$$

$$+ \text{P}(r_t^{(1)}=1, e_t^{(1)}=0/\underline{R}) \ .$$

Consider the only two possible situations: (1) $r_t^{(1)}=0$; or $r_t^{(1)}=1$. In these two situations, Rule 1 is equivalent to

Rule 1.a

$$r_t^{(1)} = 0: \ \text{P}(e_t^{(1)}=0/\underline{R}) \overset{\overset{\overset{\text{decide}}{\hat{I}_t=0}}{\gtrless}}{\underset{\underset{\hat{I}_t=0}{\text{decide}}}{}} \text{P}(e_t^{(1)}=1/\underline{R}) \qquad (3.102)$$

Rule 1.b

$$r_t^{(1)}=1: \ \text{P}(e_t^{(1)}=1/\underline{R}) \overset{\overset{\overset{\text{decide}}{\hat{I}_t=0}}{\gtrless}}{\underset{\underset{\hat{I}_t=1}{\text{decide}}}{}} \text{P}(e_t^{(1)}=0/\underline{R}) \qquad (3.103)$$

Instead of the above rule, one could employ the following _error-based_ rule:

RULE 2:

$$P(e_t^{(1)} = 0/\underline{R}) \quad \overset{\substack{\text{decide} \\ \hat{e}_t^{(1)} = 0}}{\underset{\substack{\text{decide} \\ \hat{e}_t^{(1)} = 1}}{\gtrless}} \quad P(e_t^{(1)} = 1/\underline{R}) \qquad (3.104)$$

and set $\hat{I}_t = r_t^{(1)} \oplus \hat{e}_t^{(1)}$.

It is easily seen that for all possible cases, Rule 1 (as implemented by either Rules 1.a or 1.b) will give the same estimate $\hat{I}_t$ as will Rule 2. Thus both methods of estimating $I_t$ are equally valid in the systematic code case considered. That is, selecting $\hat{I}_t$ to be that value of $I_t$ which maximizes $P(I_t/\underline{R})$ is equivalent to setting $\hat{I}_t$ equal to the value $r_t^{(1)} \oplus \hat{e}_t^{(1)}$ where $\hat{e}_t^{(1)}$ is that value of $e_t^{(1)}$ which maximizes $P(e_t^{(1)}/\underline{R})$.

Basically the practical reason for trying to use a syndrome based decoder instead of a MAP data bit decoder using $\underline{R}$ is that the implementation might be easier. The syndrome-based estimator portion estimates errors with a single syndrome $\underline{Z}$ containing information on errors only followed by a mod 2 addition to obtain $\hat{I}_t$ in the systematic case. On the other hand the MAP Data bit estimator must work with $\underline{R}$ which contains information on both data and errors.

The following chapter will pursue MAP error bit decoding based on syndromes under the dual restrictions of finite decoding delay _and_ finite syndrome vector length.

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## Chapter 4

## SUBOPTIMAL ERROR ESTIMATION

In this chapter certain suboptimal error estimation techniques are derived which may be more easily implemented than the optimal techniques of Chapter 3. Specifically, ROMMAP ans SUBMAP, two suboptimal approximations to the optimal real time (fini'e delay) MAP error bit estimator of Section 3.3 will be considered. Both use a precomputed look-up table of error bit estimates addressed by syndrome vector values. It is felt that such a finite delay estimation problem best represents the case to be encountered in practice. In the following, the restrictions on the ROMMAP/SUBMAP technique will be:

(1)  only a specified finite delay $\Delta$ is permitted before an error bit decision must be made; and

(2)  only a finite length syndrome vector $\underline{Z}$ (or received vector $\underline{R}$) is available to the decoder.

The ROMMAP technique will be first shown to be a particular average of the optimal finite delay estimator. Next the actual methods of computing MAP error bit estimates given either $\underline{R}$ or $\underline{Z}$ will be derived. Lastly, SUBMAP, a computationally convenient approximation  to the suboptimal ROMMAP approach will be discussed.

## 4.1 ROMMAP Derived from Optimal Finite Delay Error Decoding

First consider the optimal finite delay MAP error bit decoding of Section 3.3, and compare it to the suboptimal ROMMAP error decoding approach mentioned in Section 2.4.2. The optimal finite delay decoding attempts to make a MAP estimate of an error bit $e_{t-\Delta}$ using the computed syndrome bits up to the current time t but allowing only a finite storage of syndrome bits in such a decision. However it incorporates the influence of syndrome bits prior to t-L in making this MAP decision. ROMMAP decoding, on the other hand, essentially ignores the syndrome bits prior to t-L. In other words, compare

maximum over all values of $e_{t-\Delta}$ $\left\{ P(e_{t-\Delta}/z_{(1,t)}) \right\}$ versus maximum over all values of $e_{t-\Delta}$ $\left\{ P(e_{t-\Delta}/z_{(t-\Delta, t)}) \right\}$

      OPTIMAL           ROMMAP     (4.1)

where for now we have taken the delay $\Delta$ from the current time t to be _equal_ to the storage delay length L.

Note first that the required probability for the ROMMAP and the optimal can (via Bayes rule) be written as follows.

ROMMAP:

$$P(e_{t-\Delta}/z_{(t-\Delta, t)}) = \frac{P(e_{t-\Delta}, z_{(t-\Delta, t)})}{P(z_{(t-\Delta, t)})} \qquad (4.2)$$

OPTIMAL:

$$P(e_{t-\Delta}/z_{(1,t)}) = \frac{P(e_{t-\Delta}, z_{(1,t)})}{P(z_{(1,t)})}$$

$$= \frac{P(e_{t-\Delta}, z_{(1,t-\Delta-1)}, z_{(t-\Delta,t)})}{P(z_{(1,t-\Delta-1)}, z_{(t-\Delta,t)})}$$

$$= \frac{P(z_{(t-\Delta,t)}/e_{t-\Delta}, z_{(1,t-\Delta-1)})P(e_{t-\Delta}, z_{(1,t-\Delta-1)})}{P(z_{(t-\Delta,t)}/z_{(1,t-\Delta-1)})P(z_{(1,t-\Delta-1)})} \tag{4.3}$$

Now consider $P(e_{t-\Delta}, z_{(1,t-\Delta-1)})$. Since $e_{t-\Delta}$ affects $z_{t-\Delta}$ and $\nu$ subsequent syndrome bits $z_{t-\Delta+1}, \ldots, z_{t-\Delta+\nu}$, $e_{t-\Delta}$ is statistically independent of $z_{(1,t-\Delta-1)}$ (the past syndromes). Therefore,

$$P(e_{t-\Delta}, z_{(1,t-\Delta-1)}) = P(e_{t-\Delta})P(z_{(1,t-\Delta-1)}) \tag{4.4}$$

and the Optimal case probability of interest can be expressed as:

$$P(e_{t-\Delta}/z_{(1,t)}) = \frac{P(z_{(t-\Delta,t)}/e_{t-\Delta}, z_{(1,t-\Delta-1)})P(e_{t-\Delta})P(z_{(1,t-\Delta-1)})}{P(z_{(t-\Delta,t)}/z_{(1,t-\Delta-1)})P(z_{(1,t-\Delta-1)})} \tag{4.5}$$

Cancelling equal terms of numerator and denominator in the optimal case yields the following.

OPTIMAL:

$$P(e_{t-\Delta}/z_{(t-\Delta,t)}) = \frac{P(e_{t-\Delta})P(z_{(t-\Delta,t)}/e_{t-\Delta}, z_{(1,t-\Delta-1)})}{P(z_{(t-\Delta,t)}/z_{(1,t-\Delta-1)})} \tag{4.6}$$

ROMMAP:

$$P(e_{t-\Delta}/z_{(t-\Delta,t)}) = \frac{P(e_{t-\Delta})P(z_{t-\Delta,t)}/e_{t-\Delta})}{P(z_{(t-\Delta,t)})} \tag{4.7}$$

Note that (4.6) and (4.7) are similar in form. The basic difference is that the optimal quantities are conditioned on the <u>past</u> syndrome bits $z_{(1, t-\Delta-1)}$ which are not available to the ROMMAP approach.

Equations (4.6) and (4.7) can simply be rearranged to give the following forms.

OPTIMAL:

$$P(e_{t-\Delta}/z_{(1, t)}) = \frac{P(e_{t-\Delta}, z_{(t-\Delta, t)}/z_{(1, t-\Delta-1)})}{P(z_{(t-\Delta, t)}/z_{(1, t-\Delta-1)})} \qquad (4.8)$$

ROMMAP:

$$P(e_{t-\Delta}/z_{(t-\Delta, t)}) = \frac{P(e_{t-\Delta}, z_{(t-\Delta, t)})}{P(z_{(t-\Delta, t)})} \qquad (4.9)$$

Now by taking the formal average of the Optimal Eq. (4.8) or Eq. (4.3) over $z_{(1, t-\Delta-1)}$, one obtains the ROMMAP equation (4.9). That is

$$\sum_{z_1} \cdots \sum_{z_{t-\Delta-1}} P(e_{t-\Delta}/z_{(1, t)}) = \sum_{z_1} \cdots \sum_{z_{t-\Delta-1}} \frac{P(e_{t-\Delta}, z_{(1, t-\Delta-1)}, z_{(t-\Delta, t)})}{P(z_{(1, t-\Delta-1)}, z_{(t-\Delta, t)})}$$

$$\text{Average of OPTIMAL} = \frac{P(e_{t-\Delta}, z_{(t-\Delta, t)})}{P(z_{(t-\Delta, t)})}$$

$$= P(e_{t-\Delta}/z_{(t-\Delta, t)}) \quad \text{ROMMAP} \qquad (4.10)$$

Equation (4.10) shows that the ROMMAP technique is actually

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just what one would obtain from the optimal approach if one was required to average over all "past" (i.e. times less than $t-\Delta$) syndrome bits.

## 4.2 <u>MAP Estimate of $e_{t-\Delta}^{(1)}$ given Z</u>

In the ROMMAP approach to syndrome decoding of the example K-5 rate 1/2 systematic code, an estimate $\hat{e}_{t-\Delta}^{(1)}$ is obtained based on the computed finite syndrome vector $\underline{Z}$. It is assumed that sucesssive syndrome bits $\{Z_t^{(i)}\}$ are computed and stored in associated shift registers of length L bits. The syndrome vector available in this example is thus:

$$\underline{Z} = (Z_t^{(2)}, Z_{t-1}^{(2)}, \ldots, Z_{t-L}^{(2)})$$

This vector would appear in storage elements as follows.

| $Z_{t-L}^{(2)}$ | $Z_{t-L+1}^{(2)}$ | $\ldots$ | $Z_{t-1}^{(2)}$ | $Z_t^{(2)}$ |

←———— L stages of storage ————→

←——————— $\underline{Z}$ vector ———————→
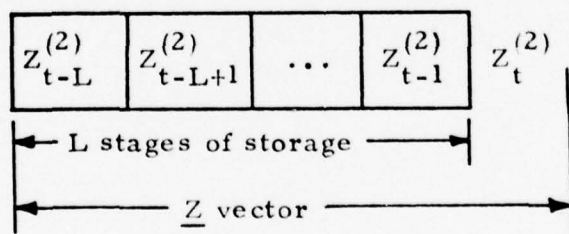
It is useful to write out part of the syndrome equation set for this example using the convention of Eq. (2.18) of Section 2.3.

$$Z_i^{(2)} = e_i^{(2)} + \sum_{h=0}^{\nu} G_{1,h}^{(2)} e_{i-h}^{(1)} \quad , \quad t-L \leq i \leq t, \quad \nu = 4 \qquad (4.11)$$

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This is done below in Eq. ( 4.12 ) for $L > \Delta$ and can be used to gauge the complexity of a particular choice of L and the effect of feedback on this same set of equations. Shown is the particular set for decoding delay $\Delta = 5$ (i.e. estimate error bit $e_{t-5}^{(1)}$). The dotted line designates the upper "triangle" of error bits which would have been previously estimated and fed back if error estimate feedback were employed.

$$
\underline{Z}
\begin{cases}
Z_{t-L} = e_{t-L}^{(2)} + G_0^{(2)} e_{t-L}^{(1)} + G_1^{(2)} e_{t-L-1}^{(1)} + G_2^{(2)} e_{t-L-2}^{(1)} + G_3^{(2)} e_{t-L-3}^{(1)} + G_4^{(2)} e_{t-L-4}^{(1)} \\[4pt]
Z_{t-L+1} = e_{t-L+1}^{(2)} + G_0^{(2)} e_{t-L+1}^{(1)} + G_1^{(2)} e_{t-L}^{(1)} + G_2^{(2)} e_{t-L-1}^{(1)} + G_3^{(2)} e_{t-L-2}^{(1)} + G_4^{(2)} e_{t-L-3}^{(1)} \\[4pt]
\quad\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\[4pt]
Z_{t-5} = e_{t-5}^{(2)} + G_0^{(2)} e_{t-5}^{(1)} + G_1^{(2)} e_{t-6}^{(1)} + G_2^{(2)} e_{t-7}^{(1)} + G_3^{(2)} e_{t-8}^{(1)} + G_4^{(2)} e_{t-9}^{(1)} \\[4pt]
Z_{t-4} = e_{t-4}^{(2)} + G_0^{(2)} e_{t-4}^{(1)} + G_1^{(2)} e_{t-5}^{(1)} + G_2^{(2)} e_{t-6}^{(1)} + G_3^{(2)} e_{t-7}^{(1)} + G_4^{(2)} e_{t-8}^{(1)} \\[4pt]
Z_{t-3} = e_{t-3}^{(2)} + G_0^{(2)} e_{t-3}^{(1)} + G_1^{(2)} e_{t-4}^{(1)} + G_2^{(2)} e_{t-5}^{(1)} + G_3^{(2)} e_{t-6}^{(1)} + G_4^{(2)} e_{t-7}^{(1)} \\[4pt]
Z_{t-2} = e_{t-2}^{(2)} + G_0^{(2)} e_{t-2}^{(1)} + G_1^{(2)} e_{t-3}^{(1)} + G_2^{(2)} e_{t-4}^{(1)} + G_3^{(2)} e_{t-5}^{(1)} + G_4^{(2)} e_{t-6}^{(1)} \\[4pt]
Z_{t-1} = e_{t-1}^{(2)} + G_0^{(2)} e_{t-1}^{(1)} + G_1^{(2)} e_{t-2}^{(1)} + G_2^{(2)} e_{t-3}^{(1)} + G_3^{(2)} e_{t-4}^{(1)} + G_4^{(2)} e_{t-5}^{(1)} \\[4pt]
Z_t = e_t^{(2)} + G_0^{(2)} e_t^{(1)} + G_1^{(2)} e_{t-1}^{(1)} + G_2^{(2)} e_{t-2}^{(1)} + G_3^{(2)} e_{t-3}^{(1)} + G_4^{(2)} e_{t-4}^{(1)}
\end{cases}
$$

(4.12)

Given an $\underline{Z}$ vector we want to find the maximum a posteriori probability estimate of $e_{t-\Delta}^{(1)}$ which in the systematic code example can then be mod 2 added to $r_{t-\Delta}^{(1)}$ to give $\hat{I}_{t-\Delta}$, the estimated $(t-\Delta)$ th information bit.

$\hat{e}_{t-\Delta}^{(1)}$ is thus set equal to that value of $e_{t-\Delta}^{(1)}$ which maximizes the following probability:

$$\underset{\substack{\text{max over all} \\ \text{values v of} \\ e_{t-\Delta}^{(1)}}}{} \left\{ \Pr(e_{t-\Delta}^{(1)} = v / \underline{Z}) = \Pr(e_{t-\Delta}^{(1)} = v / (Z_{t-L}^{(2)}, \dots, Z_t^{(2)}) \right\} \quad (4.13)$$

But this can be expressed as:

$$\underset{\substack{\text{max} \\ \text{over} \\ v}}{} \left\{ \Pr(e_{t-\Delta}^{(1)} = v / \underline{Z}) \right\} = \underset{\substack{\text{max} \\ \text{over} \\ v}}{} \left\{ \frac{\Pr(\underline{Z} / e_{t-\Delta}^{(1)} = v) \Pr(e_{t-\Delta}^{(1)} = v)}{\Pr(\underline{Z})} \right\}$$

$$= \underset{\substack{\text{max} \\ \text{over} \\ v}}{} \left\{ \Pr(\underline{Z} / e_{t-\Delta}^{(1)} = v) \Pr(e_{t-\Delta}^{(1)} = v) \right\} \quad (4.14)$$

This decision rule requires the computation of $\Pr(\underline{Z} / e_{t-\Delta}^{(1)} = v)$.

In the <u>binary</u> case the rule simplifies to the following:

$$\Pr(\underline{Z} / e_{t-\Delta}^{(1)} = 1) \Pr(e_{t-\Delta}^{(1)} = 1) \underset{\substack{\text{decide} \\ \hat{e}_{t-\Delta}^{(1)} = 0}}{\overset{\substack{\text{decide} \\ \hat{e}_{t-\Delta}^{(1)} = 1 \\ \gtrless}}{}} \Pr(\underline{Z} / e_{t-\Delta}^{(1)} = 0) \Pr(e_{t-\Delta}^{(1)} = 0) \quad (4.15)$$

For example, if the left side of Eq. (4.15) is greater than the right side, then decide $\hat{e}_{t-\Delta}^{(1)} = 1$; if the left side is less than the right side, then decide $\hat{e}_{t-\Delta}^{(1)} = 0$.

Since no assumption has been made of an orthogonal code, $\Pr(\underline{Z} / e_{t-\Delta} = 1)$ for example can be expanded in terms of each of the conditional probabilities of the other error bits and averaged over the probability of occurrence of each error bit value for the

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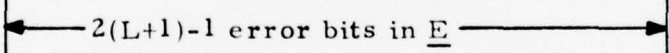
memoryless channel. For the BSC, the $Pr(e_k=1) = p$ and $Pr(e_k=0) = q = 1-p$ where p is the BSC crossover probability. That is, the first step in the expansion would be:

$$Pr(\underline{Z}/e_{t-\Delta}^{(1)}=1) = Pr(\underline{Z}/e_{t-\Delta}^{(1)}=1, e_{t-\Delta+1}^{(1)}=1)Pr(e_{t-\Delta+1}^{(1)}=1)$$
$$+ Pr(\underline{Z}/e_{t-\Delta}^{(1)}=1, e_{t-\Delta+1}^{(1)}=0)Pr(e_{t-\Delta+1}^{(1)}=0) \qquad (4.16)$$

This expansion would be continued for both sides of the decision rule until all conditional probabilities for all error bits appearing in syndrome bit equation set for the specified $|\underline{Z}|$ were considered.

An error vector can be defined as in Section 2.3 as follows.

$$\underline{E} \overset{\Delta}{=} (e_{t-L}^{(2)}, e_{t-L}^{(1)}, \ldots, e_{t-\Delta-1}^{(1)}, e_{t-\Delta+1}^{(1)}, \ldots, e_{t}^{(2)}, e_{t}^{(1)}) \qquad (4.17)$$

$$\longleftarrow 2(L+1)-1 \text{ error bits in } \underline{E} \longrightarrow$$

(Note: $e_{t-\Delta}^{(1)}$ is not included in $\underline{E}$.)

The Maximum A posteriori Error Bit Decision Rule then is

$$p \sum_{\substack{\text{all } \underline{E} \\ \text{satisfying} \\ \text{this } \underline{Z} \text{ for} \\ e_{t-\Delta}^{(1)}=1}} p^{w(\underline{E})} q^{2(L+1)-1-w(\underline{E})} \underset{\substack{\text{decide} \\ \hat{e}_{t-\Delta}^{(1)}=0}}{\overset{\substack{\text{decide} \\ \hat{e}_{t-\Delta}^{(1)}=1}}{\gtrless}} q \sum_{\substack{\text{all } \underline{E} \\ \text{satisfying} \\ \text{this } \underline{Z} \text{ for} \\ e_{t-\Delta}^{(1)}=0}} p^{w(\underline{E})} q^{2(L+1)-1-w(\underline{E})} \qquad (4.18)$$

For example, if the left side of Eq. (4.18) is greater than the right side, decide $e_{t-\Delta}^{(1)} = 1$; if the left side is less than the right side, decide $e_{t-\Delta}^{(1)} = 0$.

For a given $E_b/N_0$ (and therefore a given BSC crossover probability $(Pr(e_k^{(i)} = 1))$ the above computation can be carried out for each possible $\underline{Z}$ vector in the set $\{\underline{Z}\}$ of $2^{(L+1)} = 2^{LSYN}$ syndrome vectors. The resultant decisions $e_{t-\Delta}^{(1)}(\underline{Z})$ can be stored in a ROM addressed by its associated $\underline{Z}$. Also stored could be the actual decision confidence $p_1(\underline{Z})$ where

$$p_1(\underline{Z}) \stackrel{\Delta}{=} p \sum_{\substack{\text{all } \underline{E} \text{ satisfying} \\ \text{this } \underline{Z} \text{ with } e_{t-\Delta}^{(1)} = 1}} q^{w(\underline{E})} q^{2(L+1)-1-w(\underline{E})} . \qquad (4.19)$$

This storage of the set $\{\underline{Z}\}$ each with its $e_{t-\Delta}^{(1)}(\underline{Z})$ is the ROMMAP - the Read Only Memory Map from a computed syndrome vector $\underline{Z}$ to its corresponding Maximum A posteriori Error Bit Estimate $e_{t-\Delta}^{(1)}(\underline{Z})$. Such a look-up table incorporated in a decoder enables error correction with essentially no decoder computations.

## 4.3 MAP Estimate of $e_{t-\Delta}^{(1)}$ Given $\underline{R}$

It is instructive to now temporarily examine just what it would take to make a ROMMAP-like (i.e., a look-up table) error estimator using the actual received vector $\underline{R}$ instead of a computed syndrome vector $\underline{Z}$. This diversion will demonstrate why the syndrome based approach of Section 4.2 is so attractive.

In basic MAP data bit decoding of a systematic convolutional code,

a decision or estimate $\hat{I}_t$ of the most probable information bit $I_t$ is made by finding that value of $I_t$ such that

$$Pr(\hat{I}_t/\underline{R}) = \begin{array}{c} \text{max} \\ \text{over all} \\ \text{values of} \\ I_t \end{array} \left\{ Pr(I_t/\underline{R}) \right\} \qquad (4.20)$$

Here $\underline{R}$ is some selected segment of received bits (a finite number) e.g.,

$$\underline{R} = (r_{t-L}^{(1)}, r_{t-L+1}^{(1)}, \ldots, r_t^{(1)}, r_{t-L}^{(2)}, \ldots, r_t^{(2)}) \qquad (4.21)$$

in our standard binary systematic convolutional code example.

McAdam [9] found a forward and backward recursion method for determining $\hat{I}_t$ given the total received vector. It might be useful to see what a ROMMAP approach to finding the most likely $I_t$ given a finite segment $\underline{R}$ would give for a binary rate $1/n$ systematic code when equilikely data inputs are assumed. That is, assume a finite decoding delay $\Delta$ and storage for a finite segment $\underline{R}$ of the received vector.

Recall that $I_t = r_t^{(1)} - e_t^{(1)} = r_t^{(1)} \oplus e_t^{(1)}$ (over GF(2)). Then from Section 3.5

(most probable $I_t$/given $r_t^{(1)}$) = $r_t^{(1)}$-(most probable $e_t^{(1)}$/given $r_t^{(1)}$).
$$(4.22)$$

Note that $r_t^{(1)}$ is available and known while $\hat{e}_t^{(1)}$ must be estimated. Also $r_t^{(1)}$ is one component of $\underline{R}$. Therefore

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$$\hat{I}_t = \begin{array}{c} \text{that } I_t \\ \text{maximizing} \end{array} \left\{ Pr(I_t/\underline{R}) \right\} = r_t^{(1)} - \begin{array}{c} \text{that } e_t^{(1)} \\ \text{maximizing} \end{array} \left\{ Pr(e_t^{(1)}/\underline{R}) \right\} \tag{4.23}$$

Since in the systematic code case $r_t^{(1)}$ is known, finding the maximum a posteriori estimate of $I_t$ given $\underline{R}$ can be accomplished by finding the maximum a posteriori estimate of $e_t^{(1)}$ given $\underline{R}$ and subtracting $\hat{e}_t^{(1)}$ from $r_t^{(1)}$.

Now concentrate on finding that value of $e_t^{(1)}$ which maximizes $Pr(e_t^{(1)}/\underline{R})$. However,

$$Pr(e_t^{(1)}=v/\underline{R}) = \frac{Pr(\underline{R}/e_t^{(1)}=v)Pr(e_t^{(1)}=v)}{Pr(\underline{R})} \tag{4.24}$$

where $Pr(e_t^{(1)}=1) = p$ and $Pr(e_t^{(1)}=0) \simeq 1-p = q$ for the BSC. $\underline{R}$ is given as received and $Pr(\underline{R}/e_t^{(1)})$ must be computed. This can be done using the same approach as for the case of $Pr(\underline{Z}/e_{t-\Delta}^{(1)})$ discussed in the ROMMAP determination by noting that

$$r_t^{(j)} = c_t^{(j)} + e_t^{(j)} \quad 1 \le j \le n \tag{4.25}$$

where

$$C_t^{(j)} = \begin{cases} I_t & \text{if } 1 \le j \le k \\ \displaystyle\sum_{\ell=1}^{k} \sum_{h=0}^{\nu} G_{\ell h} I_{t-h} & \text{if } k < j \le n. \end{cases} \tag{4.26}$$

Or, in the specific case of $k=1$ (only one info bit per block), $\nu=4$, rate 1/2 systematic code of the example this becomes:

$$r_i^{(1)} = C_i^{(1)} + e_i^{(1)} = I_i + e_i^{(1)}$$

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$$r_i^{(2)} = C_i^{(2)} + e_i^{(2)} = \sum_{h=0}^{\nu} G_{1h}^{(2)} I_{i-h} + e_i^{(2)} , \quad t-L \leq i \leq t .$$ (4.27)

If we have L bits of storage for the (j) $r_i$, $1 \leq j \leq 2$, received

bit streams, our available $\underline{R}$ is:
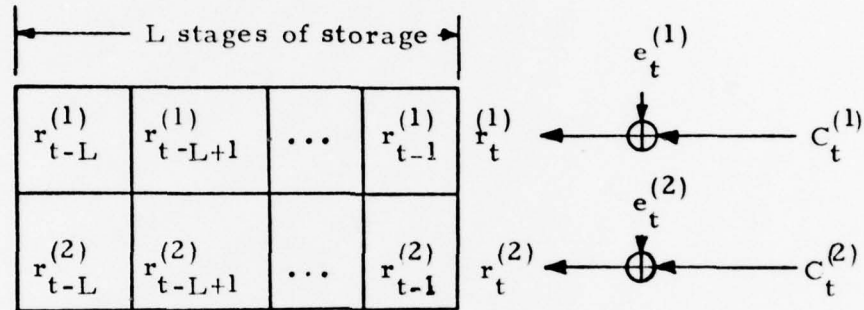


To show a comparison between the set of syndrome equations of

the ROMMAP approach (Eq. 4.12) with the corresponding set of

equations for estimating error bits from an $\underline{R}$ vector, the comparable

$\underline{received\ bit\ equations}$ for the example rate 1/2 code is given below

(Eq. ( 4.28 ) for arbitrary L.

$$r_{t-L}^{(1)} = e_{t-L}^{(1)} + I_{t-L}$$

$$r_{t-L}^{(2)} = e_{t-L}^{(2)} + G_0^{(2)}I_{t-L} + G_1^{(2)}I_{t-L-1} + G_2^{(2)}I_{t-L-2} + G_3^{(2)}I_{t-L-3} + G_4^{(2)}I_{t-L-4}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$r_{t-5}^{(1)} = e_{t-5}^{(1)} + I_{t-5}$$

$$r_{t-5}^{(2)} = e_{t-5}^{(2)} + G_0^{(2)}I_{t-5} + G_1^{(2)}I_{t-6} + G_2^{(2)}I_{t-7} + G_3^{(2)}I_{t-8} + G_4^{(2)}I_{t-9}$$

$$r_{t-4}^{(1)} = e_{t-4}^{(1)} + I_{t-4}$$

$$r_{t-4}^{(2)} = e_{t-4}^{(2)} + G_0^{(2)}I_{t-4} + G_1^{(2)}I_{t-5} + G_2^{(2)}I_{t-6} + G_3^{(2)}I_{t-7} + G_4^{(2)}I_{t-8}$$

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$$r_{t-3}^{(1)} = e_{t-3}^{(1)} + I_{t-3}$$

$$r_{t-3}^{(2)} = e_{t-3}^{(2)} + G_0^{(2)}I_{t-3} + G_1^{(2)}I_{t-4} + G_2^{(2)}I_{t-5} + G_3^{(2)}I_{t-6} + G_4^{(2)}I_{t-7}$$

$$r_{t-2}^{(1)} = e_{t-2}^{(1)} + I_{t-2}$$

$$r_{t-2}^{(2)} = e_{t-2}^{(2)} + G_0^{(2)}I_{t-2} + G_1^{(2)}I_{t-3} + G_2^{(2)}I_{t-4} + G_3^{(2)}I_{t-5} + G_4^{(2)}I_{t-6}$$

$$r_{t-1}^{(1)} = e_{t-1}^{(1)} + I_{t-1}$$

$$r_{t-1}^{(2)} = e_{t-1}^{(2)} + G_0^{(2)}I_{t-1} + G_1^{(2)}I_{t-2} + G_2^{(2)}I_{t-3} + G_3^{(2)}I_{t-4} + G_4^{(2)}I_{t-5}$$

$$r_t^{(1)} = e_t^{(1)} + I_t$$

$$r_t^{(2)} = e_t^{(2)} + G_0^{(2)}I_t + G_1^{(2)}I_{t-1} + G_2^{(2)}I_{t-2} + G_3^{(2)}I_{t-3} + G_4^{(2)}I_{t-4}$$

$$(4.28)$$

A total of $n(L+1)$ bits are received and held in $\underline{R}$ where $n=2$ in this particular example. As in the ROMMAP computations we can expand $\Pr(\underline{R}/e_{t-\Delta}^{(1)})$ in terms of the conditional probabilities for all available error $\underline{\text{and}}$ information bits. That is we want that value of $e_{t-\Delta}^{(1)}$ which maximizes equation (4.24). But this is equivalent to finding that $e_{t-\Delta}^{(1)}$ value which produces the

$$\begin{array}{c}\text{max over}\\\text{all values}\\ e_{t-\Delta}^{(1)}\end{array} \left\{ \Pr(\underline{R}/e_{t-\Delta}^{(1)})\Pr(e_{t-\Delta}^{(1)}) \right\} \qquad (4.29)$$

This value can be computed from equation (4.24) by expanding $\Pr(\underline{R}/e_{t-\Delta}^{(1)})$ repeatedly until all conditional probabilities for both error and information bits have been incorporated. The first step would be:

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$$Pr(\underline{R}/e_{t-\Delta}^{(1)}=v) = Pr(\underline{R}/e_{t-\Delta}^{(1)}=v, x_k=1)Pr(x_k=1)$$

$$+ Pr(\underline{R}/e_{t-\Delta}^{(1)}=v, x_k=0)Pr(x_k=0) \qquad (4.30)$$

where $x_k$ is some particular info or error bit. At each step in the
expansion the new bit introduced ($x_k$) takes on another identity -
either some information bit (e.g. $I_{t-L+k}$) or some error bit
(e.g. $e_{t-L+k}^{(i)}$). When working with $\underline{R}$ it is here that the presence
of information bits must be accounted for in the computation; while
in the syndrome ROMMAP case, all dependence upon information
bits was removed when syndrome bits were computed.

Using the definitions,

$$\text{Error vector} = \underline{E} \triangleq [e_{t-L}^{(2)}, e_{t-L}^{(1)}, \ldots, e_{t-\Delta-1}^{(1)}, e_{t-\Delta+1}^{(1)}, \ldots, e_t^{(2)}, e_t^{(1)}]$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.31)$

$$\text{Information vector} = \underline{I} \triangleq [I_{t-L-4}, \ldots, I_t] \qquad (4.32)$$

provides a notationally simpler form for the expansion. Note that
$e_{t-\Delta}^{(1)}$ is not included as a component of the $\underline{E}$ vector. Also if $x_k$ is an
error bit then $Pr(x_k=1) = p$ and $Pr(x_k=0) = 1-p$, while if $x_k$ is an
information bit then $Pr(x_k=1) = Pr(x_k=0) = \frac{1}{2}$ (i.e., assuming equi-
likely source inputs).

The quantity $Pr(\underline{R}/e_{t-\Delta}^{(1)}=1)Pr(e_{j-\Delta}^{(1)}=1)$ for example can now be
written as

$$\Pr(\underline{R}/e^{(1)}_{t-\Delta}{=}1)\Pr(e^{(1)}_{t-\Delta}{=}1){=}p\sum_{\substack{\text{all }\underline{EI}\\ \text{satisfying this}\\ \underline{R}\text{ with }e^{(1)}_{t-\Delta}{=}1}}p^{w(\underline{E})}q^{2(L+1)-1-w(\underline{E})}P(I{=}1)^{w(\underline{I})}P(I{=}0)^{(L+5-w(\underline{I}))}.$$

$$\tag{4.33}$$

But with equilikely inputs $P(I{=}1) = P(I{=}0) = \frac{1}{2}$ and since $e^{(1)}_{t-\Delta}$ is 1 or 0, the new form of the decision rule is:

$$\Pr(\underline{R}/e^{(1)}_{t-\Delta}{=}1)\Pr(e^{(1)}_{t-\Delta}{=}1)$$

$$= p\sum_{\substack{\text{all }\underline{EI}\\ \text{satisfying this}\\ \underline{R}\text{ with }e^{(1)}_{t-\Delta}{=}1}}p^{w(\underline{E})}q^{2(L+1)-1-w(\underline{E})}(\tfrac{1}{2})^{w(\underline{I})}(\tfrac{1}{2})^{(L+5)-w(\underline{I})}$$

$$\begin{array}{c}\text{decide}\\ \widehat{e}^{(1)}_{t-\Delta}{=}1\\ >\\ <\\ \widehat{e}^{(1)}_{t-\Delta}{=}0\end{array}$$

$$q\sum_{\substack{\text{all }\underline{EI}\\ \text{satisfying this}\\ \underline{R}\text{ with }e^{(1)}_{t-\Delta}{=}1}}p^{w(\underline{E})}q^{2(L+1)-1-w(\underline{E})}(\tfrac{1}{2})^{w(\underline{I})}(\tfrac{1}{2})^{(L+5)-w(\underline{I})}$$

$$= \Pr(\underline{R}/e^{(1)}_{j-\Delta}{=}0)\Pr(e^{(1)}_{j-\Delta}{=}0)\,. \tag{4.34}$$

For equilikely inputs the contribution of the information bits to each side of the above rule is the same, so they can be dropped from the rule to obtain

$$p \sum_{\substack{\text{all } \underline{EI} \\ \text{satisfying this} \\ \underline{R} \text{ with } e_{t-\Delta}^{(1)} = 1}} p^{w(\underline{E})} q^{2(L+1)-1-w(\underline{E})} \underset{\substack{\text{decide} \\ \hat{e}_{t-\Delta}^{(1)} = 0}}{\overset{\substack{\text{decide} \\ \hat{e}_{t-\Delta}^{(1)} = 1}}{\underset{<}{>}}} q \sum_{\substack{\text{all } \underline{EI} \\ \text{satisfying this} \\ \underline{R} \text{ with } e_{t-\Delta}^{(1)} = 0}} p^{w(\underline{E})} q^{2(L+1)-1-w(\underline{E})} \; .$$

(4.35)

Comparing the above rule Eq. (4.35) with that for the ROMMAP estimate of $e_{t-\Delta}^{(1)}$ (Eq. (4.18)), one observes that the quantities being summed are the same but the conditions over which the sum is taken are not identical. In the ROMMAP case, the sum is over Error Vectors $\underline{E}$ satisfying the current $\underline{Z}$. In Eq. (4.35) above, the sum is over Error _and_ Info vectors $\underline{EI}$ satisfying the current $\underline{R}$. Again the problem of considering the information bits exists when one estimates $e_{t-\Delta}^{(1)}$ based on the set of received bits $\underline{R}$.

This decision rule can also be written as

$$p \sum_{\text{all } \underline{E}} N_{E1} p^{w(\underline{E})} q^{2(L+1)-1-w(\underline{E})} \underset{\substack{\text{decide} \\ \hat{e}_{t-\Delta}^{(1)} = 0}}{\overset{\substack{\text{decide} \\ \hat{e}_{t-\Delta}^{(1)} = 1}}{\underset{<}{>}}} q \sum_{\text{all } \underline{E}} N_{E0} p^{w(\underline{E})} q^{2(L+1)-1-w(\underline{E})}$$

(4.36)

where   $N_{E1}$ = number of $\underline{I}$ vectors such that $\underline{EI}$ satisfy current $\underline{R}$ with $e_{t-\Delta}^{(1)} = 1$

and   $N_{E0}$ = number of $\underline{I}$ vectors such that $\underline{EI}$ satisfy current $R$ with $e_{t-\Delta}^{(1)} = 0$ .

At each step in the computation of the above sum over all $\underline{E}$, the following applies:

(1)  $\underline{R}$ is known from the received bit streams.

(2)  $\underline{E}$ is known for this particular step, and

(3)  we need to compute the number of $\underline{I}$ vectors (out of a total of $2^{L+5}$ possible $\underline{I}$ vectors) which when substituted along with this particular $\underline{E}$ into the "$\underline{R}$ equations" will satisfy them for the current $\underline{R}$.

Since $\underline{R}$ is given and $\underline{E}$ is known for this particular step, $\underline{R}-\underline{E}$ either equals a code word or it does not. It is assumed here that L (the number of stages of storage for each received bit stream) is greater than $L_{min}$ where $L_{min}$ is a constraint length-determined number sufficient to guarantee that each possible input bit stream is encoded into a unique "codeword." With this assumption, if $\underline{R}-\underline{E}$ is not a codeword then $N_{E1}$ (or $N_{E0}$ as the case may be) is equal to zero. If $\underline{R}-\underline{E}$ is a codeword then it is a unique codeword and only one of the possible $\underline{I}$ vectors satisfies the $\underline{R}$ equations for this particular $\underline{E}$.

To simplify the following make the definition:

$$f(\underline{E}) \triangleq p^{w(\underline{E})} q^{2(L+1)-1-w(\underline{E})}. \tag{4.37}$$

The decision rule then can be written as:

$$p \sum_{\substack{\text{all } \underline{EI} \\ \text{satisfying this} \\ \underline{R} \text{ with } e_{t-\Delta}^{(1)}=1}} f(\underline{E}) \underset{\text{decide "0"}}{\overset{\text{decide "1"}}{\underset{<}{>}}} q \sum_{\substack{\text{all } \underline{EI} \\ \text{satisfying this} \\ \underline{R} \text{ with } e_{t-\Delta}^{(1)}=0}} f(\underline{E}) \cdot \quad (4.38)$$

Again to simplify notation let us define the set of "codewords" $\{C_k\}$ to be those $2(L+1)$-tuples which are output by the encoder in rssponse to the $2^{L+5}$ possible input data streams under consideration. Also, for those $\underline{E}$ which are considered by the decision rule sums (i.e. the $\underline{E}$ whose $N_{E1}$ or $N_{E0}$ equal one) we can apply

$$\underline{E} = \underline{R} \oplus C_k \quad \text{for some } C_k \in \{C_k\}. \quad (4.39)$$

The decision rule then can be written as:

$$p \sum_{\substack{\text{all } \underline{E=C_k \oplus R} \\ \text{for all } C_k \in \{C_k\} \\ \text{with } e_{t-\Delta}^{(1)}=1}} f(\underline{E}) \underset{\substack{\text{decide} \\ \hat{e}_{t-\Delta}^{(1)}=0}}{\overset{\substack{\text{decide} \\ \hat{e}_{t-\Delta}^{(1)}=1}}{\underset{<}{>}}} q \sum_{\substack{\text{all } \underline{E=C_k \oplus R} \\ \text{for all } C_k \in \{C_k\} \\ \text{with } e_{t-\Delta}^{(1)}=0}} f(\underline{E}) \quad \cdot \quad (4.40)$$

Eq. (4.40) above then equivalently directs the decoder to consider all possible error patterns $\underline{E}$ which could have mapped any codeword $\underline{C}_k$ into the actual received vector $\underline{R}$ and weigh them proportionately by their probability of occurrence $f(\underline{E})$ to make the maximum a posteriori decision on this particular error bit $e_{t-\Delta}^{(1)}$. Thus the additional complexity that arises by attempting to estimate error bits $e_{t-\Delta}$ from the received vector $\underline{R}$ without using syndromes

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has been demonstrated. Consequently, the bulk of this work considers error bit estimation on _syndromes_ only.

## 4.4 SUBMAP Approximation to ROMMAP

In Section 4.2 the ROMMAP decision rule for estimating the error bit $e_{t-\Delta}$ for K=5 rate $\frac{1}{2}$ systematic code example with finite delay $\Delta$ and only a finite syndrome vector $\underline{Z}$ was shown to be:

ROMMAP Decision Rule

$$
p \sum_{\substack{\text{all } \underline{E} \\ \text{satisfying this} \\ \underline{Z} \text{ with } e_{t-\Delta}^{(1)}=1}} p^{w(\underline{E})} q^{2(L+1)-1-w(\underline{E})} \underset{\substack{\text{decide} \\ \hat{e}_{t-\Delta}^{(1)}=0}}{\overset{\substack{\text{decide} \\ \hat{e}_{t-\Delta}^{(1)}=1 \\ >}}{\underset{<}{}}} q \sum_{\substack{\text{all } \underline{E} \\ \text{satisfying this} \\ \underline{Z} \text{ with } e_{t-\Delta}^{(1)}=0}} p^{w(\underline{E})} q^{2(L+1)-1-w(\underline{E})}
$$

$$(4.41)$$

This ROMMAP decision rule can be implemented by a read only memory addressed by $\underline{Z}$ and containing the single error bit estimate for each $\underline{Z}$ given by the rule. To create such a look-up table would require considering all error patterns $\{\underline{E}'\}$ where $\underline{E}' \overset{\Delta}{=} (\underline{E}, e_{t-\Delta})$ (thus $\underline{E}'$ contains all error bits including the one to be estimated). In such a binary case this means considering $2^{|\underline{E}'|}$ separate error patterns where $|\underline{E}'| \overset{\Delta}{=}$ the number of error bits in this overall error vector. Such an exponential growth in computational complexity is a basic drawback to any look-up table decoding approach.

This drawback can be mitigated somewhat by taking advantage of the non-equilikely nature of the error bit values. On a practical

binary symmetric channel, the $\Pr(e_t^{(i)}=1) \ll \Pr(e_t^{(i)}=0)$. Thus only "lightweight" error vectors (those having few 1's) have a significant probability of occurring. Therefore, if instead of considering all possible error vectors $\{\underline{E}'\}$ as dictated by the ROMMAP decision rule, one was to consider only a lightweight subset of $\{\underline{E}'\}$, the computational complexity of creating the look-up table could be reduced. The resultant SUBMAP look-up table would not be significantly different than that of the full ROMMAP since the neglected contributions to the sums due to "heavy" $\underline{E}$ in 4.41 would be extremely small. Thus the decisions found for either table would be almost the same.

To see the complexity reduction in look-up table computation, consider the K=5 rate $\frac{1}{2}$ code. If a syndrome vector of LSYN = 15 syndrome bits is considered, the look-up table in either case would contain $2^{15} = 2^{LSYN}$ locations. The ROMMAP computation would require considering $2^{33+1} = 2^{34} = 17,179,869,210$ error vectors $\underline{E}'$. If instead one only includes error vectors of weight 5 or less, then the SUBMAP computation requires considering $\binom{34}{0} + \binom{34}{1} + \binom{34}{2} + \binom{34}{3} + \binom{34}{4} + \binom{34}{5} = 331,212$ error vectors for a complexity reduction by a factor of approximately 51,870.

For completeness the SUBMAP computation rule for the previous rate $\frac{1}{2}$ binary example may be written as:

## SUBMAP Decision Rule

$$p \sum_{\substack{\text{all } \underline{E} \\ \text{satisfying this} \\ \underline{Z} \text{ such that } w(\underline{E}) \leq 4 \\ \text{with } e_{t-\Delta}^{(1)}}} p^{w(\underline{E})} q^{2(L+1)-1-w(\underline{E})} \underset{\substack{\text{decide} \\ e_{t-\Delta}^{(1)}=0}}{\overset{\substack{\text{decide} \\ e_{t-\Delta}^{(1)}=1 \\ >}}{\underset{<}{\gtrless}}} q \sum_{\substack{\text{satisfying this} \\ \underline{Z} \text{ such that} \\ w(\underline{E}) \leq 5}} p^{w(\underline{E})} q^{2(L+1)-1-w(\underline{E})}.$$

<div align="right">(4.42)</div>

Due to the rapid exponential growth in computation, in this work the full ROMMAP computation was carried out only for a K = 5 rate $\frac{1}{2}$ code for the case $|\underline{Z}| = 6 = LSYN$ syndrome bits. All subsequent work for longer constraint length codes or more syndrome bits concentrated on the SUBMAP approach. It is shown in Chapter 5 that there is available a very tight upper bound on performance using the SUBMAP approach which enables the accurate prediction of performance simultaneously with the creation of the SUBMAP look-up table.

A specific comment is in order on the relationship between ROMMAP, SUBMAP, and conventional feedback decoder approaches. The conventional feedback decoder as described by Heller [16] uses as the error bit estimate that value of $e_{t-\Delta}^{(1)}$ appearing in the <u>minimum weight error vector</u> that could have caused the given syndrome vector $\underline{Z}$. This means that the many-to-one mapping of $\{E\}$ vectors

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into syndrome vectors $\{Z\}$ is approximated by a one-to-one mapping

using only <u>one</u> of the lightest weight error vectors $E$ . This is con-

trasted with the ROMMAP approach in which the <u>entire</u> many-to-one

mapping is considered in making the error bit estimate and with the

SUBMAP approach in which a significant portion of the many-to-one

mapping is used.

Thus the three approaches can be ranked by how close they

approximate the true finite delay-finite syndrome MAP estimate:

| <u>Approach</u> | <u>Mapping Considered</u> | $\hat{e}$ is: |
|---|---|---|
| ROMMAP | Entire | MAP estimate |
| SUBMAP | Subset of entire | Approximate MAP estimate |
| Conventional Feedback | Single $E$ | Grosser approximation to MAP estimate |

Note that in all three approaches the size of the decoder look-up

table is the same ($2^{|Z|}$ in the binary case). Thus one is logically led

to assume that, given a specification on the maximum $|Z|$ (and

consequently the maximum size of the look-up table), it would be

advisable to use look-up table entries computed by considering as

large a subset of $\{E\}$ vectors as possible thus approaching the

ROMMAP or true Finite Delay-Finite Syndrome MAP error bit

estimate.

Chapter 5 demonstrates that this is indeed the case since both

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computed performance and simulation results show that the SUBMAP

decoders outperform conventional feedback (i.e. minimum weight $\underline{E}$)

decoders. In many cases a SUBMAP decoder even without feedback

will perform better than a conventional feedback decoder.

Chapter 5

PERFORMANCE PREDICTION AND SIMULATION RESULTS

In Chapter 3 several theoretically optimal error bit decoding approaches were derived. Chapter 4 took one of these, optimal finite delay error bit decoding, and exhibited two suboptimal approximations, the ROMMAP and SUBMAP decoders. In this chapter decoder performance prediction is analyzed for these two decoders when implemented for typical codes. It is shown that an actual probability of bit error can be computed for the ROMMAP decoder while a tight upper bound on performance of the SUBMAP decoder can be obtained as the look-up table is created. A lower bound based on an idealized first error probability is developed and shown to give a lower limit to continued performance improvement for a given code and decoding delay. Also the performance improvement possible both by changing to a longer constraint length code and by increasing $|\underline{Z}|$ for a given code is displayed. Lastly, Monte Carlo SUBMAP simulation results for two codes using various $|\underline{Z}|$ values with and without feedback are presented and compared with conventional feedback decoder performance.

5.1 ROMMAP Creation and Predicted Performance

In this section the actual computation of a ROMMAP look-up table to implement a binary MAP finite delay-finite syndrome error

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decoding rule is qualitatively described along with the manner of computing the actual probability of bit error for the ROMMAP decoder. This look-up table is addressed by the current syndrome vector value $\underline{Z}$. Defining LSYN $\overset{\Delta}{=} |\underline{Z}|$ = the number of syndrome bits in the syndrome vector means that the size of the ROMMAP table must be $2^{\text{LSYN}}$. Each addressed storage location in the table must contain either a 0 or 1 value corresponding to the MAP error bit estimate $\hat{e}_{t-\Delta}^{(1)}$ determined by the rule. This value will sometimes be denoted as $\hat{e}_{t-\Delta}^{(1)}(\underline{Z})$ in the following explanation for a binary rate $\frac{1}{2}$ systematic code.

### 5.1.1 ROMMAP Creation

It should be specifically noted that the two entities of interest to our decoder for a rate $\frac{1}{2}$ systematic code are:

(1) $\underline{Z} = (Z_t^{(2)}, Z_{t-1}^{(2)}, \ldots, Z_{t-L}^{(2)}$ (provided in real time by the syndrome former and syndrome bit shift register); and

(2) $\hat{e}_{t-\Delta}^{(1)}$ (the error bit to be estimated by reference to the ROMMAP look-up table).

Here L is the "syndrome bit delay parameter" indicating that L+1 consecutive syndrome bits comprise the syndrome vector $\underline{Z}$. It is always assumed that $Z_t^{(2)}$ is the latest syndrome bit output by the syndrome former. Also $\Delta$ is the "error bit delay parameter" indicating that $\Delta+1$ noisy information bits $r_j^{(1)}$, $j = t, t-1, \ldots, t-\Delta$, are delayed and available to be corrected. This means that at time t,

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when $r_t^{(1)}$ and $Z_t^{(2)}$ become available to the decoder, the ROMMAP decoder will produce $\hat{e}_{t-\Delta}^{(1)}$ by "looking up" the value $\hat{e}_{t-\Delta}^{(1)}(\underline{Z})$ in the ROMMAP table and mod 2 add it to $r_{t-\Delta}^{(1)}$ to give the information bit estimate

$$\hat{I}_{t-\Delta} = r_{t-\Delta}^{(1)} + \hat{e}_{t-\Delta}^{(1)}(\underline{Z})$$

Consequently the "decoding delay" is $\Delta$.

The distinction between $\Delta$ and L is worth making clear since, although syndrome decoding schemes such as conventional feedback decoding usually implement $L \approx \Delta$, there is no inherent reason why the decoding delay $\Delta$ must equal the syndrome bit delay L. In the work that follows $L \geq \Delta$.

For convenience, the actual implementations discussed in this and later sections were restricted to binary systematic codes. For such codes the total ROMMAP table approach may be mechanized in a hardwired device which only has to determine if the current syndrome vector $\underline{Z}$ belongs to the subset of syndromes such that $\hat{e}_{t-\Delta}^{(1)}(\underline{Z}) = 1$. That is, the contents of a ROMMAP table actually map the $2^{LSYN}$ syndrome vectors into two points, $(0, 1)$. A study of the redundancy in the syndromes-to-decision mapping was not pursued in the work but would seem to be a possible means of speeding decisions when compared to a slow look-up table search.

For later reference, the ROMMAP decision rule for a systematic rate $\frac{1}{2}$ code is:

$$p \sum_{\substack{\text{all } \underline{E} \\ \text{satisfying this} \\ \underline{Z} \text{ with } e_{t-\Delta}^{(1)}=1}} p^{w(\underline{E})}(1-p)^{LEN-w(\underline{E})} \mathop{\gtrless}_{\substack{\text{decide} \\ \hat{e}_{t-\Delta}^{(1)}=0}}^{\substack{\text{decide} \\ \hat{e}_{t-\Delta}^{(1)}=1}} (1-p) \sum_{\substack{\text{all } \underline{E} \\ \text{satisfying this} \\ \underline{Z} \text{ with } e_{t-\Delta}^{(1)}=0}} p^{w(\underline{E})}(1-p)^{LEN-w(\underline{E})}$$

$$(5.2)$$

where: $\underline{E}$ is the vector of all error bits entering into the set of

LSYN syndrome equations other than the bit $e_{t-\Delta}^{(1)}$ to be estimated;

LEN is the length of $\underline{E}$; $\underline{Z}$ is the current syndrome vector of length

$|\underline{Z}|$ = LSYN; and p is the BSC crossover probability. Note that

Eq. (5.2) is merely the decision rule resulting from considering

only the numerator of the ROMMAP probability in Eq. 4.14 and

specializing to a rate $\frac{1}{2}$ binary case.

For the relatively short constraint length binary examples of

ROMMAP and SUBMAP decoders considered here, the first step

in implementation is the creation of the look-up table, i.e. the

mapping from syndrome vectors $\underline{Z}$ to a particular error bit estimate

$\hat{e}_{t-\Delta}^{(1)}$. In the ROMMAP case this mapping is done by deterministically

generating all $2^{LEN}$ error patterns, substituting each into the right

hand side of the appropriate set of syndrome equations with the error

bit to be estimated set to 0 and then to 1. For each of the $2^{LEN}$

error patterns this gives two syndromes (equivalent to each of the

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$2^{LEN+1}$ overall error patterns mapping to one syndrome). A data structure consisting of a double vector of probabilities indexed by the $2^{LSYN}$ possible syndrome values was used to accumulate contributions to $P(\underline{Z}/e_{t-\Delta}^{(1)}=1)$ and $P(\underline{Z}/e_{t-\Delta}^{(1)}=0)$ as all error patterns were processed. Once all error patterns had been processed, the data structure entries are multiplied by p and 1-p respectively to yield the terms of the decision rule (5.2), for each $\underline{Z}$. By cycling through all $\underline{Z}$ locations in the data structure the MAP error bit decision $\hat{e}_{t-\Delta}^{(1)}(\underline{Z})$ at each $\underline{Z}$ can be made based on these terms and recorded in a third level of the data structure to give a triple vector. This third level is the ROMMAP look-up table as shown in Figure 5.1.

5.1.2 Signal to Noise Ratio

One should note here that since the contributions to the $P(e_{t-\Delta}^{(1)}=1/\underline{Z})$ and $P(e_{t-\Delta}^{(1)}=0/\underline{Z})$ numerators are based on the crossover probability p of the binary symmetric channel, it is necessary to consider how this p is related to the signal to noise (SNR) level of the received signal. For a particular $\underline{E}$ vector of weight $w(\underline{E})$ for example, the contribution would be

$$p^{w(\underline{E})}(1-p)^{LEN+1-w(\underline{E})} \text{ for the syndrome when } e_{t-\Delta}^{(1)} = 0 \text{ and} \qquad (5-3a)$$

$$p^{w(\underline{E})+1}(1-p)^{LEN-w(\underline{E})} \text{ for the syndrome when } e_{t-\Delta}^{(1)} = 1 \qquad (5-3b)$$

The crossover probability p can be related back to a signal to noise ratio assuming additive white Gaussian noise and a particular

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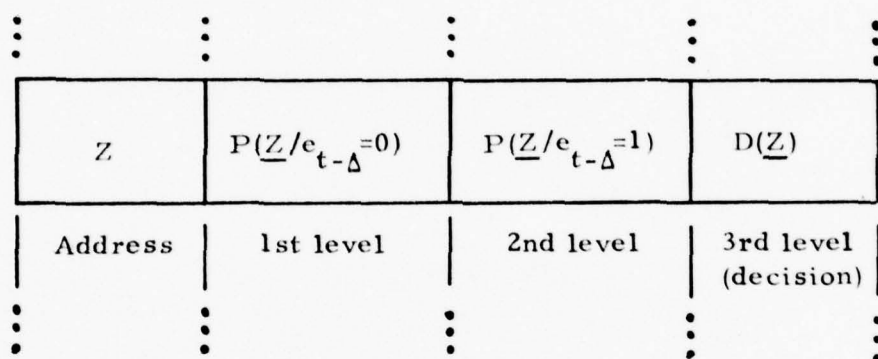| Z | $P(\underline{Z}/e_{t-\Delta}=0)$ | $P(\underline{Z}/e_{t-\Delta}=1)$ | $D(\underline{Z})$ |
|---|---|---|---|
| Address | 1st level | 2nd level | 3rd level (decision) |

Figure 5.1.   Triple Level Data Structure for ROMMAP Creation.

modulation. Thus any ROMMAP look-up table has associated with it a "SNR design value" corresponding to the p used in the probability contribution computations.

In the case of 180 degree binary phase shift keying (BPSK) modulation assumed throughout this work, one can easily relate the BSC crossover probability p to SNR level. The additive white Gaussian noise causes the received channel symbol level v to have the conditional distributions shown in Figure 5.2 assuming proper receiver gain adjustment to provide unit variance as noted by McAdam $[9]$. Note that in Fig. 5.2 the mean of the conditional distributions is $\pm\sqrt{2E_s/N_0}$ and the cross-hatched area can be used to compute the BSC crossover probability p.

Thus

$$p = \int_{v=0}^{v=+\infty} P(v/0)dv = \int_{v=0}^{v=+\infty} \frac{1}{\sqrt{2\pi}} \exp^{-\frac{1}{2}(v+\sqrt{2E_s/N_0})^2} dv.$$

(5.4)

With $x = v + \sqrt{2E_s/N_0}$ one has

$$p = \int_{x=+\sqrt{2E_s/N_0}}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp^{-\frac{1}{2}x^2} dx$$

and with $u = x/\sqrt{2}$ one obtains

$$p = \frac{1}{\sqrt{\pi}} \int_{u=\sqrt{E_s/N_0}}^{\infty} \exp^{-u^2} du = \frac{1}{2}\left(\frac{2}{\sqrt{\pi}} \int_{u=\sqrt{E_s/N_0}}^{+\infty} e^{-u^2} du = \frac{1}{2} \operatorname{erfc}(\sqrt{E_s/N_0})\right)$$
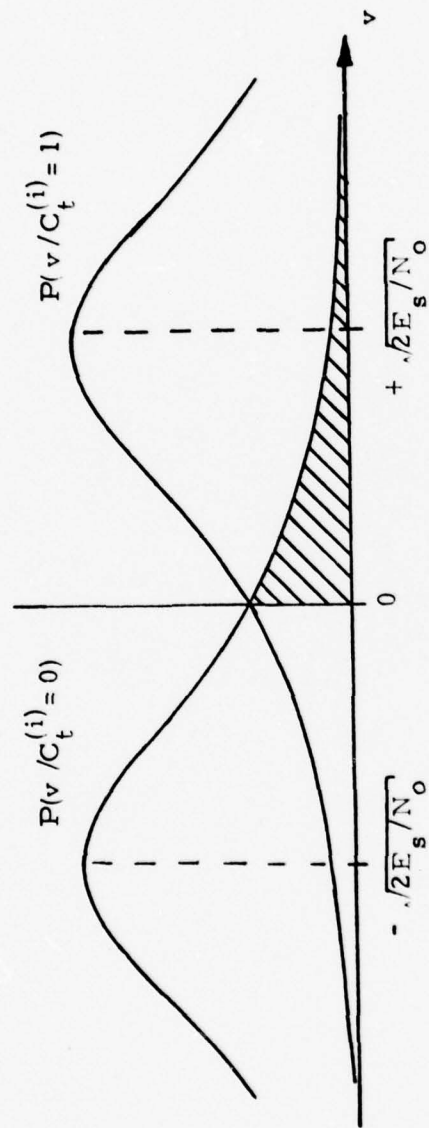
(5.5)

where

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Figure 5.2 Conditional Densities of Received Channel Symbol Level
for BPSK Modulation and AWGN

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp^{-t^2} dt.$$

This can also be expressed as

$$p = \frac{1}{2}[1 - \text{erf}(E_s/N_0)] \quad \text{where } \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp^{-t^2} dt \qquad (5.6)$$

as noted in Wu [15]. In the above it should be emphasized that

$E_s \triangleq$ average signal energy per transmitted bit (symbol energy)

and

$N_0$ = noise power density.

If one desires p for BPSK to be expressed in terms of the more conventional $E_b$, the uncoded signal energy per bit for a rate R code, then, using $E_s/N_0 = (E_b/N_0)R$, the BSC crossover probability becomes:

$$p = \frac{1}{2}[1 - \text{erf}(E_b R/N_0)] \qquad (5.7)$$

where R = the code rate.

Since all codes actually implemented and compared in this work were rate $R = \frac{1}{2}$, both predicted and simulated data for probability of bit error are plotted in terms of $E_s/N_0$ in db, the transmitted symbol signal to noise ratio. This is consistent with the approach of other authors (e.g. Heller [16], Schalkwijk and Vinck [13]) who tend to plot "$P_{out}$ versus $P_{in}$." However some caution must be used in comparing results of other work (e.g. Heller and Jacobs [17]) where $P_{out}$ (equivalent to $P_e$ herein) is plotted in terms of $E_b/N_0$.

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### 5.1.3  ROMMAP Probability of Bit Error

Once the ROMMAP look-up table has been computed, sufficient information is available in the aforementioned data structure to actually compute a probability of error for the error decision. The set of addresses $\{\underline{Z}\}$ of the ROMMAP table can be divided into two disjoint subsets:

$$Z_0 = \{\text{the set of all syndrome vectors for which look-up table entries } \hat{e}_{t-\Delta}(\underline{Z})=0\} \tag{5.8}$$

and

$$Z_1 = \{\text{the set of all syndrome vectors for which look-up table entries } \hat{e}_{t-\Delta}(\underline{Z})=1\}$$

Thus $Z = Z_0 \cup Z_1$ and $Z_0 \cap Z_1 = \emptyset$. The ROMMAP decoder operates by deciding:

and
$$\hat{e}_{t-\Delta} = 0 \text{ if } \underline{Z} \epsilon Z_0$$
$$\hat{e}_{t-\Delta} = 1 \text{ if } \underline{Z} \epsilon Z_1 \ . \tag{5.9}$$

To determine the probability of making a wrong error bit decision $P_e$, it is convenient to define:

$$P_{e,0} \overset{\Delta}{=} \{\text{probability of making a wrong error bit decision when } e_{t-\Delta}=0\}$$

$$= \sum_{\underline{Z} \epsilon Z_1} P(\underline{Z}/e_{t-\Delta}=0) \tag{5.10}$$

and

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$P_{e,1} \triangleq$ {probability of making a wrong error bit decision when $e_{t-\Delta}=1$}

$$= \sum_{\underline{Z} \in Z_0} P(\underline{Z}/e_{t-\Delta}=1) \qquad (5.11)$$

Averaging over both possible values of $e_{t-\Delta}$ gives the average error probability $P_e$:

$$P_e = P(e_{t-\Delta}=0)P_{e,0} + P(e_{t-\Delta}=1)P_{e,1} \qquad (5.12)$$

The quantities $P(\underline{Z}/e_{t-\Delta}=0)$ and $P(\underline{Z}/e_{t-\Delta}=1)$ appearing in (5.10) and (5.11) are the same as the quantities computed in creating the ROMMAP look-up table. Thus (5.10) and (5.11) can be evaluated for the particular code and particular LSYN value by summing over those data structure addresses $\underline{Z}$ for which $\hat{e}_{t-\Delta}(\underline{Z})=1$ and $\hat{e}_{t-\Delta}(\underline{Z})=0$ respectively. Which $\underline{Z}$ is in which subset is already flagged by the 0,1 entries at each address $\underline{Z}$. This implies that, for a particular systematic binary code, the computation necessary for the creation of a ROMMAP table also provides a theoretical value for probability of decoding error $P_e$ at the design SNR. For example Figure 5.4 shows ROMMAP $P_e$ as computed for a K=5 code versus design SNR (lower curve).

Since the actual ROMMAP performance can be calculated, the use of any Chernoff or Simple Bound approach as used in Gallagher [18] for performance prediction is not warranted. Indeed when one pursues either approach it is found that the same quantities

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$P(\underline{Z}/e_{t-\Delta}=0)$ and $P(\underline{Z}/e_{t-\Delta}=1)$ are needed for either bound evaluation so one might just as well compute $P_e$ directly as described above.

Thus is has been shown that if a look-up table is created for a ROMMAP decoder, an actual probability of bit decoding error is readily available. However, as noted in Section 4.4, the number of error vectors $\underline{E}$ that must be considered for the full ROMMAP grows exponentially with syndrome vector length, LSYN. Thus one is led to consider the SUBMAP look-up table approach and a method of predicting or bounding $P_e$ for SUBMAP decoding. This bound is considered in the following section.

## 5.2 SUBMAP Creation and Performance

Due to the exponential growth in computational complexity with syndrome vector length for the ROMMAP decoder, it is feasible to use the full ROMMAP approach only for quite short constraint lengths. Consequently it is worthwhile to consider creating a SUBMAP look-up table decoder as discussed in Section 4.4. This will be done in the following sections along with consideration of performance prediction and the relationship to conventional feedback decoding.

### 5.2.1 SUBMAP Creation

In computing a SUBMAP look-up table, the same three level data structure described above for the ROMMAP approach is used. It is loaded with probability contributions as before but only a lightweight subset of all possible $\underline{E}'$ vectors is considered. Thus

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contributions from the subset of $\underline{E}'$ vectors have been entered, the
data structure entries at each $\underline{Z}$ address are $P^*(\underline{Z}/e_{t-\Delta})$, $P^*(\underline{Z}/e_{t-\Delta})$,
and $D(\underline{Z}) = \hat{e}_{t-\Delta}^{(1)}(\underline{Z})$. The "$*$" is used to differentiate these quantities
from the corresponding ROMMAP quantities since they will be only
approximately the same. The SUBMAP entries will be less than or
equal to the ROMMAP entries. That is $P(\underline{Z}/e_{t-\Delta}) = P^*(\underline{Z}/e_{t-\Delta}) + \delta$
where $\delta$ is the difference due to the neglected "heavyweight" vectors
If the maximum weight of error vectors (MAX) in the lightweight
subset is increased toward LEN+1 the SUBMAP and ROMMAP
tables will become identical. Just how similar the performances
of the two approaches will be is explained in the following section.
For reference purposes the three level SUBMAP data structure can
be visualized as depicted in Figure 5.3.

## 5.2.2 A Tight Upper Bound On SUBMAP Performance

Since the SUBMAP approach creates a look-up table by consider-
ing only a lightweight subset of the total set of possible error vectors,
the effect on performance of the excluded error vectors must be
measured or bounded.

Consider expressing the average error probability $P_e$ for the
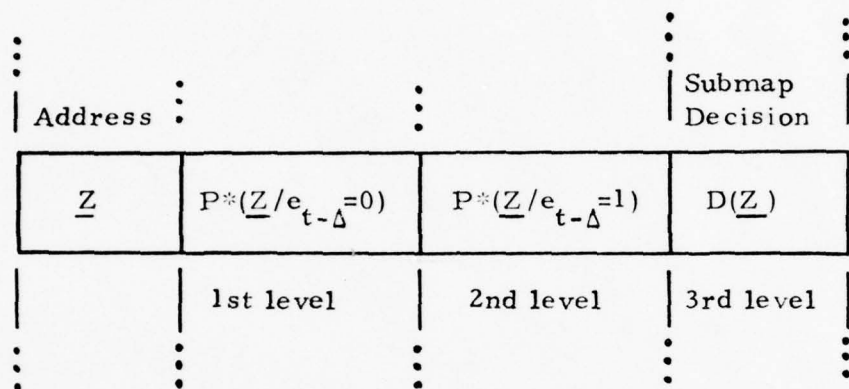SUBMAP case as follows:

Figure 5.3. Triple-level Data Structure for SUBMAP creation.

$$P_e = \sum_{\substack{\text{all } \underline{E}' \\ \text{vectors}}} P(\text{error}/\underline{E}')P(\underline{E}') = \sum_{\substack{\text{all } \underline{E}' \text{ vectors} \\ \text{considered in} \\ \text{SUBMAP}}} P(\text{error}/\underline{E}')P(\underline{E}')$$

$$+ \sum_{\substack{\text{all } \underline{E}' \text{ vectors} \\ \text{excluded from} \\ \text{SUBMAP}}} P(\text{error}/\underline{E}')P(\underline{E}') \qquad (5.13)$$

In the second summation above, the quantity $P(\text{error}/\underline{E}') \leq 1$. To upper bound $P_e$, set $P(\text{error}/\underline{E}') = 1$ <u>only</u> in the second term. In other words assume a decoding error is made with probability equal to one whenever an excluded $\underline{E}'$ occurs. This yields the bound:

$$P_e \leq \sum_{\substack{\text{all } \underline{E}' \text{ vectors} \\ \text{considered in} \\ \text{SUBMAP}}} P(\text{error}/\underline{E}')P(\underline{E}') + \sum_{\substack{\text{all } \underline{E}' \text{ vectors} \\ \text{excluded from} \\ \text{SUBMAP}}} (1)P(\underline{E}') \qquad (5.14)$$

Now for notational convenience define:

(1)  $D(\underline{Z}) = \hat{e}_{t-\Delta}(\underline{Z}) = $ the SUBMAP table 0, 1 decision addressed by $\underline{Z}$;

(2)  $\underline{Z}(\underline{E}') = $ the syndrome vector which corresponds to a particular error vector $\underline{E}'$;

(3)  $\hat{P} = \sum_{\substack{\text{all } \underline{E}' \text{ vectors} \\ \text{considered in} \\ \text{SUBMAP}}} P(\text{error}/\underline{E}')P(\underline{E}');$ and (4) $P_{\text{skip}} = \sum_{\substack{\text{all } \underline{E}' \text{ vectors} \\ \text{excluded from} \\ \text{SUBMAP}}} (1)P(\underline{E}').$

$$\qquad (5.15)$$

Thus

$$P_e \leq \hat{P} + P_{skip} \qquad (5.16)$$

Now consider $\hat{P}$. A decision mistake can be made in the following two ways when an $\underline{E}'$ vector contained in the subset considered in deriving the SUBMAP table is present:

(1)  the error vector $\underline{E}'$ is such that $e_{t-\Delta} = 1$ but $D(\underline{Z}(\underline{E})) = 0$; or

(2)  the error vector $\underline{E}'$ is such that $e_{t-\Delta} = 0$ but $D(\underline{Z}(E)) = 1$.

In case (1) the contribution to the sum due to such an $\underline{E}'$ having $e_{t-\Delta} = 1$ is

$$
\begin{aligned}
P(error/\underline{E}')P(\underline{E}') &= P(D(\underline{Z}(\underline{E}'))=0)P(\underline{E}') \\
&= (1-D(\underline{Z}(\underline{E}'))P(\underline{E}') \ . \qquad (5.17)
\end{aligned}
$$

In case (2) the contribution to the sum due to such an $\underline{E}'$ having $e_{t-\Delta} = 0$ is

$$
\begin{aligned}
P(error/\underline{E}')P(\underline{E}') &= P(D(\underline{Z}(\underline{E}'))=1)P(\underline{E}') \\
&= D(\underline{Z}(\underline{E}'))P(\underline{E}') \qquad (5.18)
\end{aligned}
$$

By the previously described method of loading the look-up table data structure:

$$P^*(\underline{Z}/e_{t-\Delta}=1) = \sum_{\substack{\text{all } \underline{E}' \text{ in SUBMAP} \\ (\text{with } e_{t-\Delta}=1) \text{ mapping} \\ \text{to this } \underline{Z}}} P(\underline{E}) \quad \text{and} \quad P^*(\underline{Z}/e_{t-\Delta}=0) = \sum_{\substack{\text{all } \underline{E}' \text{ in SUBMAP} \\ (\text{with } e_{t-\Delta}=0) \\ \text{mapping to this } \underline{Z}}} P(\underline{E})$$

$$(5.19)$$

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Therefore,

$$\hat{P} = \sum_{\text{all } \underline{Z}} (1-D(\underline{Z}))P^{*}(\underline{Z}/e_{t-\Delta}=1) + \sum_{\text{all } \underline{Z}} D(\underline{Z})P^{*}(\underline{Z}/e_{t-\Delta}=0) \quad (5.20)$$

and $\hat{P}$ can be easily calculated from the three level SUBMAP data structure entries.

Now consider $P_{skip}$. This is merely the probability that an error vector $\underline{E}'$ occurs which has not been considered in creating the SUBMAP. Such an error vector must have weight greater than MAX, the maximum weight of error vectors considered in the SUBMAP. For notational convenience let LNER = LEN+1 = the number of bits in the overall error vector $\underline{E}'$ (including the bit to be estimated). Also note that the probability of having present an error vector weight w of length LNER is

$$P_{w} = \binom{\text{LNER}}{w} p^{w}(1-p)^{\text{LNER}-w} \quad (5.21)$$

Since all $\underline{E}'$ vectors having more than MAX ones have been excluded,

$$P_{skip} = \sum_{\substack{\text{all } \underline{E}' \\ \text{excluded} \\ \text{from SUBMAP}}} P(\underline{E}) = \sum_{i=\text{MAX}+1}^{\text{LNER}} P_{i} = 1 - \sum_{i=0}^{\text{MAX}} P_{i} \quad (5.22)$$

Thus it is a straightforward matter to compute both (5.20) and (5.22) to yield an upper bound on the actual $P_{e}$ using the quantities

already available from the SUBMAP look-up table computation.

This bound will be quite tight especially at higher SNR levels where p is small thus reducing the magnitude of $P_{skip}$. Figure 5.4 shows the tightness of this upper bound as MAX is increased from 1 to 5. For the K=5, LSYN=6 decoder case shown, one notices that the bound is looser at low SNR and tighter at high SNR. Also for $MAX \geq d_{min} = 4$ there is little difference ($\leq 0.1$ db) between the theoretical ROMMAP performance and the upper bound on SUBMAP performance. However since the SUBMAP creation takes significantly less computations than the ROMMAP approach, it represents a much more feasible way to gauge performance. With the proper choice of a MAX weight considering the available minimum distance $d_{min}$ of the code, one can closely approach ROMMAP performance with a SUBMAP decoder. The predicted performance shown by the bounds in Figure 5.4 will be confirmed by simulation in Section 5.5.

5.2.3 Performance Comparison with Conventional Feedback

The SUBMAP look-up table is created by considering a lightweight subset of E' vectors specified by MAX. Included in this subset is at least one minimum weight E' vector mapping to each syndrome Z. Such a single E' vector is commonly used to make the look-up table for conventional feedback decoders (see Heller [16]). Thus, for comparison purposes, as the three level data structure is loaded to generate the SUBMAP table, one can also generate a fourth level
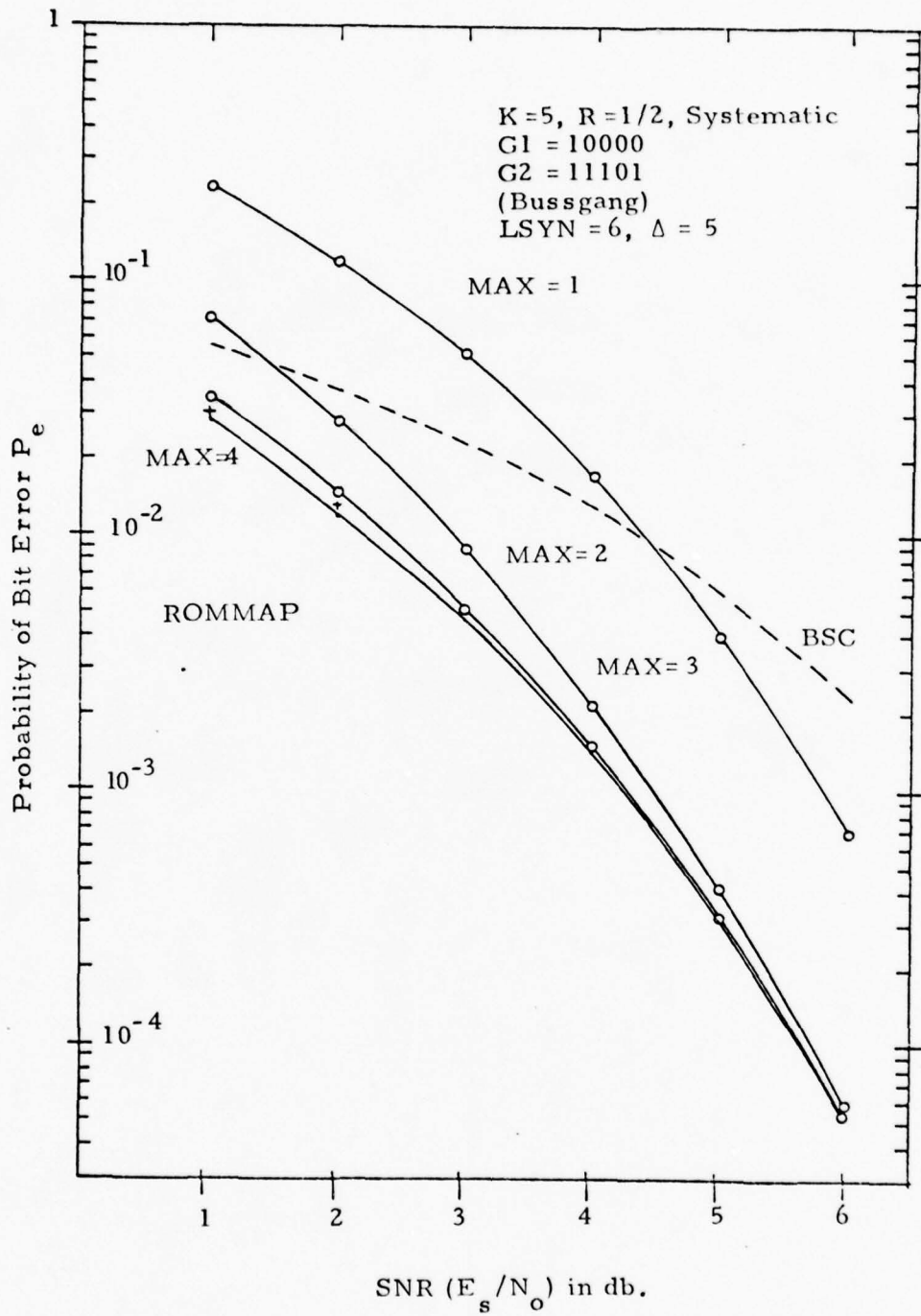
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Figure 5.4 Upper Bound on SUBMAP Performance

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containing a "minimum weight" look-up table obtained by associating

with each $\underline{Z}$ the error bit $e_{t-\Delta}^{(1)}$ found in a minimum weight $\underline{E}'$ mapping

to this $\underline{Z}$. This was done in the creation of look-up tables for the

code and LSYN combinations simulated in Section 5.5 to enable a

side by side comparison of actual SUBMAP versus actual conventional

feedback performance.

The same upper bounding arguments described in Section 5.2.2

for the SUBMAP table can be applied to the minimum weight error

vector table (conventional feedback) to give an upper bound on the

performance of a decoder using the conventional feedback table

without feedback. Figure 5.5 shows the predicted performance

bounds for both the SUBMAP and conventional minimum weight tables

for LSYN = 13 with a K = 11 code at a constant SNR level of 2 db

$(E_s/N_0)$. As expected, the SUBMAP upper bound remains lower

than the minimum weight (without feedback) upper bound as MAX is

increased to consider a larger subset of lightweight error vectors.

Both predicted SUBMAP and minimum weight bounds do not

account for the potential effect of feeding back error estimates to

correct syndromes. These predicted performance bounds give

an upper limit on error probability when the results of prior

decisions are not considered. One thus is led to hope that employing

feedback may improve SUBMAP performance as it is known to

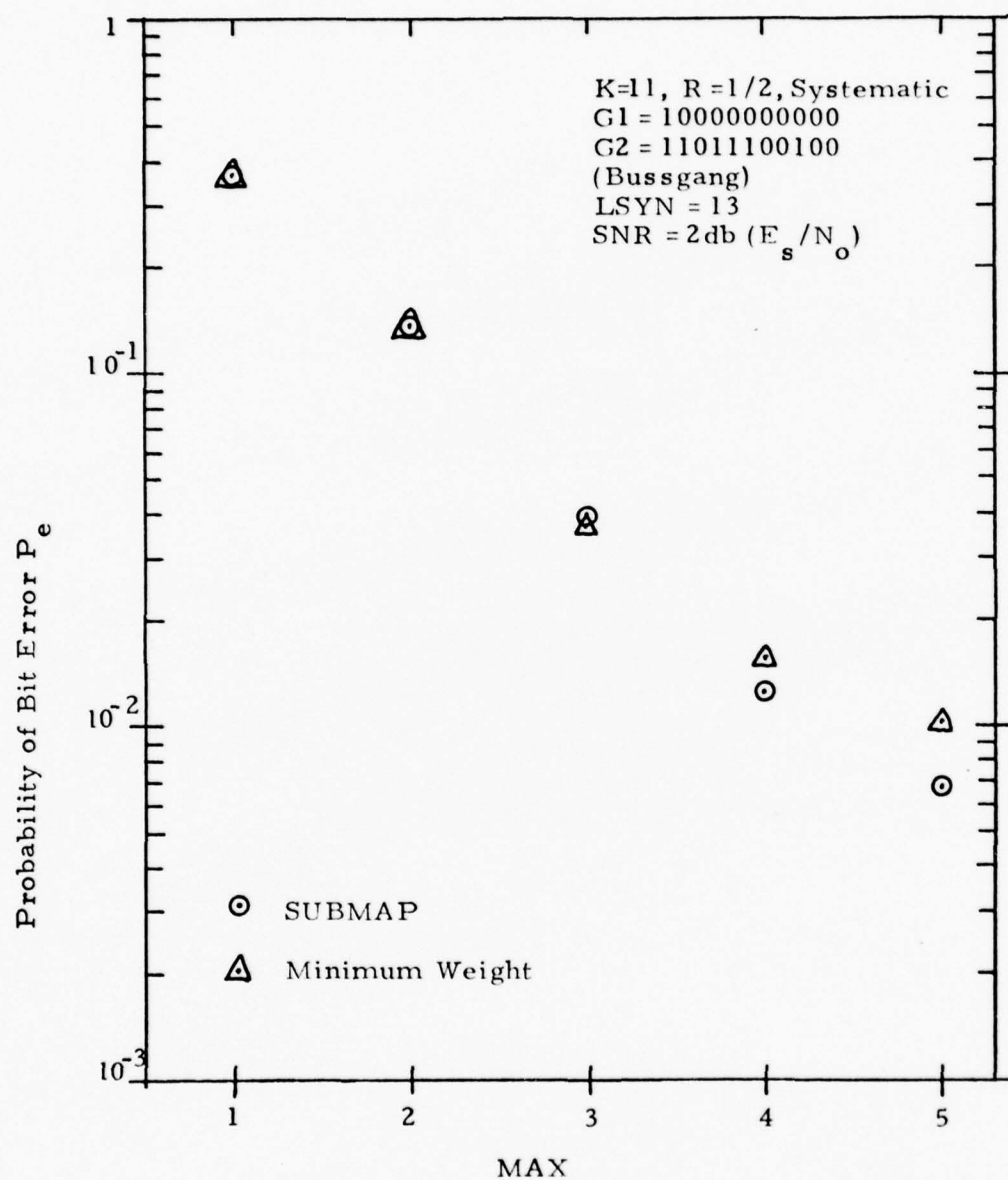improve minimum weight performance. The extent to which this is

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Figure 5.5 SUBMAP and Minimum Weight Performance
as a Function of MAX

true is shown in Section 5.5 simulation.

## 5.3 Performance Improvement with Increasing Syndrome Length

In order to demonstrate the beneficial effect of using syndrome lengths greater than one constraint length, predicted SUBMAP performance for codes of constraint length $K = 5, 7$, and $11$ were computed for several LSYN values. These predictions are given in Figures 5.6, 5.7, and 5.8 respectively and show how performance should improve as the number of syndrome bits used is increased.

From these predictions one can also observe that, for a given LSYN, substituting a code with a longer constraint length and minimum distance for a shorter constraint length code should provide better performance. LSYN is the parameter that determines the size of the SUBMAP look-up table and thus effectively the complexity of the decoder. Thus in a practical design one is led to allow the permitted level of complexity (look-up table size) determine a corresponding LSYN value and use a code having a constraint length as near to this LSYN as possible.

Specifically, using a longer constraint length code (e.g. $K=11$) provides better performance than using a shorter constraint length code (e.g. $K=5$) for the same number of syndrome bits (e.g. LSYN=11). This is to be expected due to the greater minimum distance ($d_{min}=7$) of the $K=11$ code versus $d_{min}=4$ for the $K=5$ code.
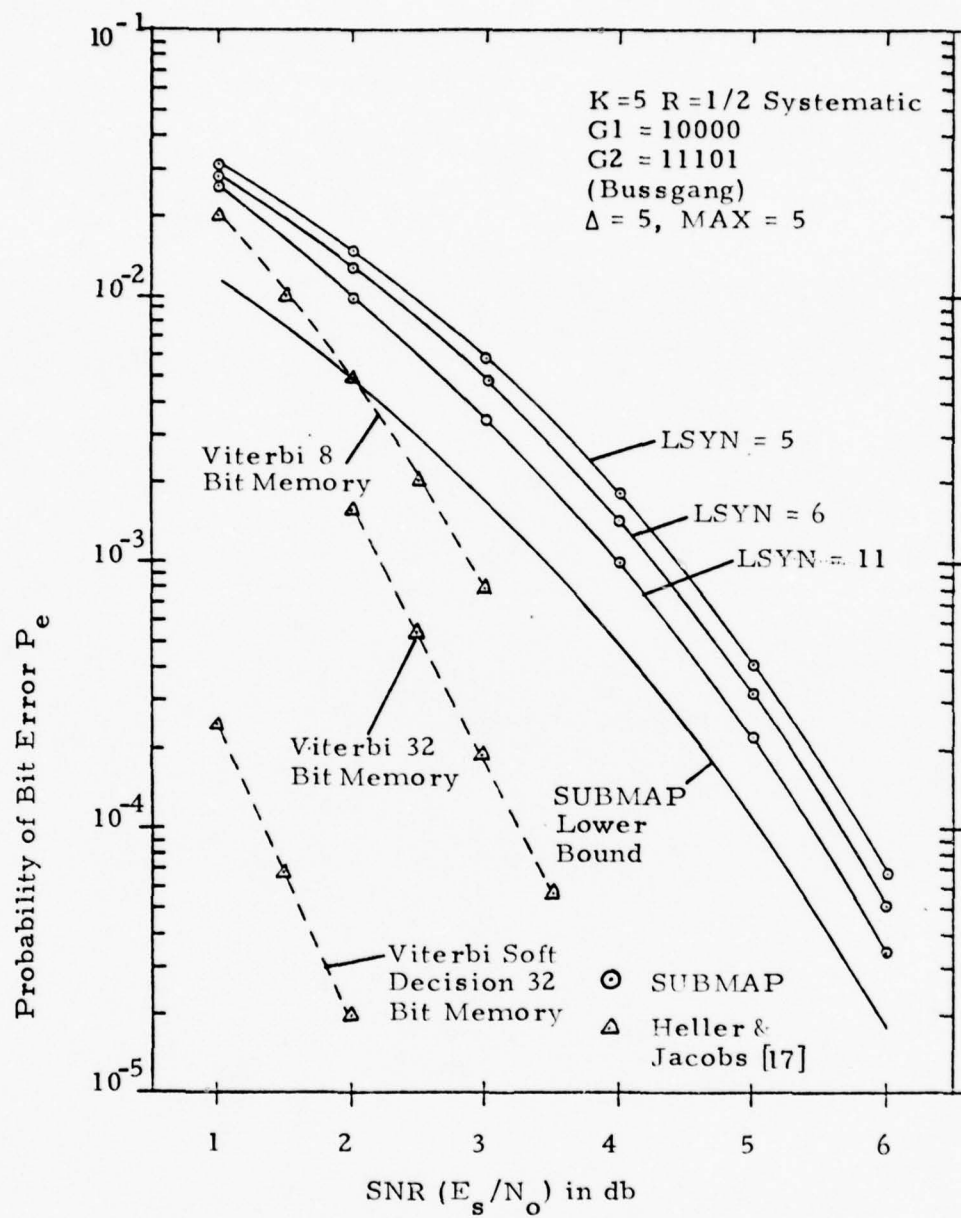
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Figure 5.6  SUBMAP Predicted Performance, K = 5
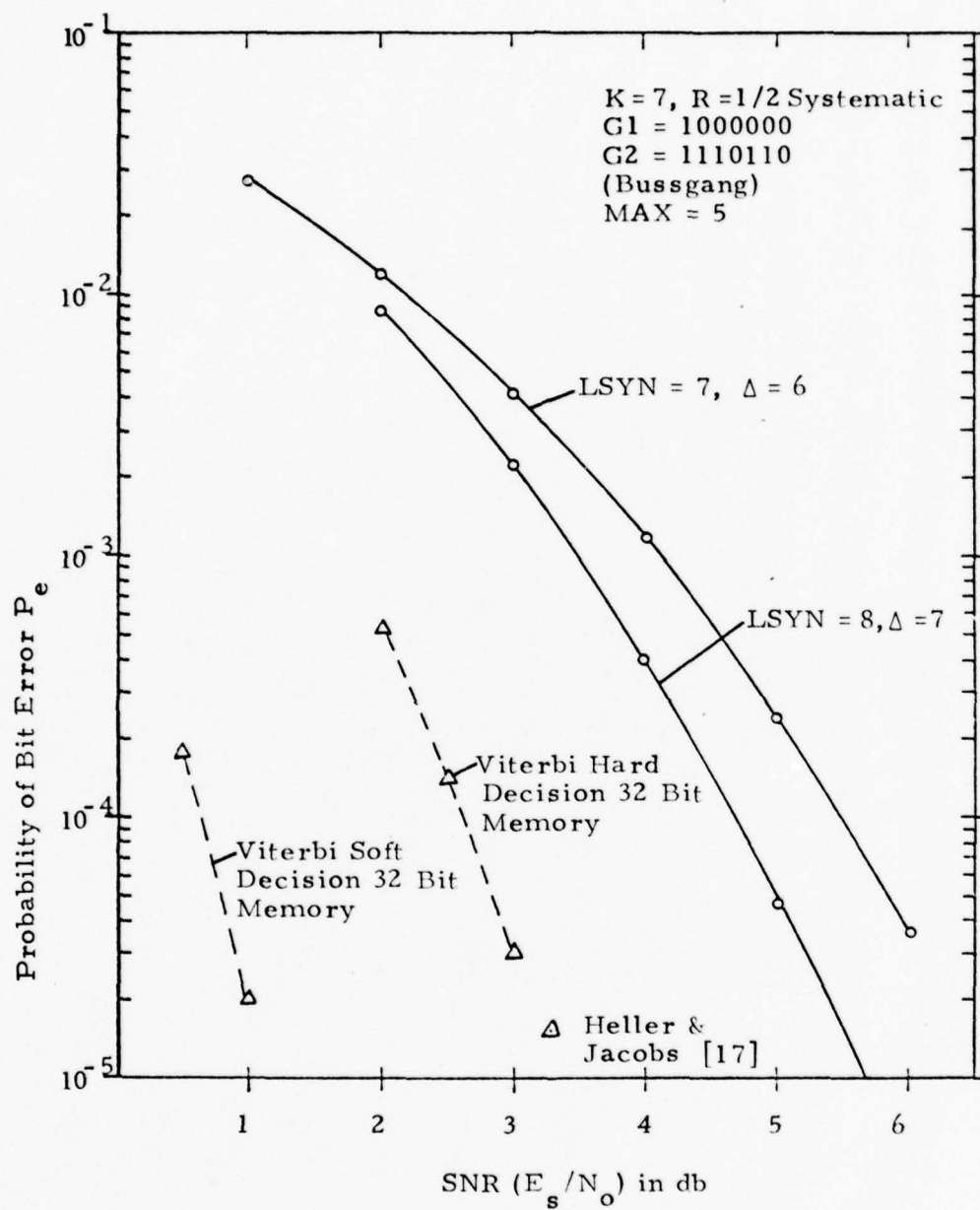
Figure 5.7  SUBMAP Predicted Performance, K=7

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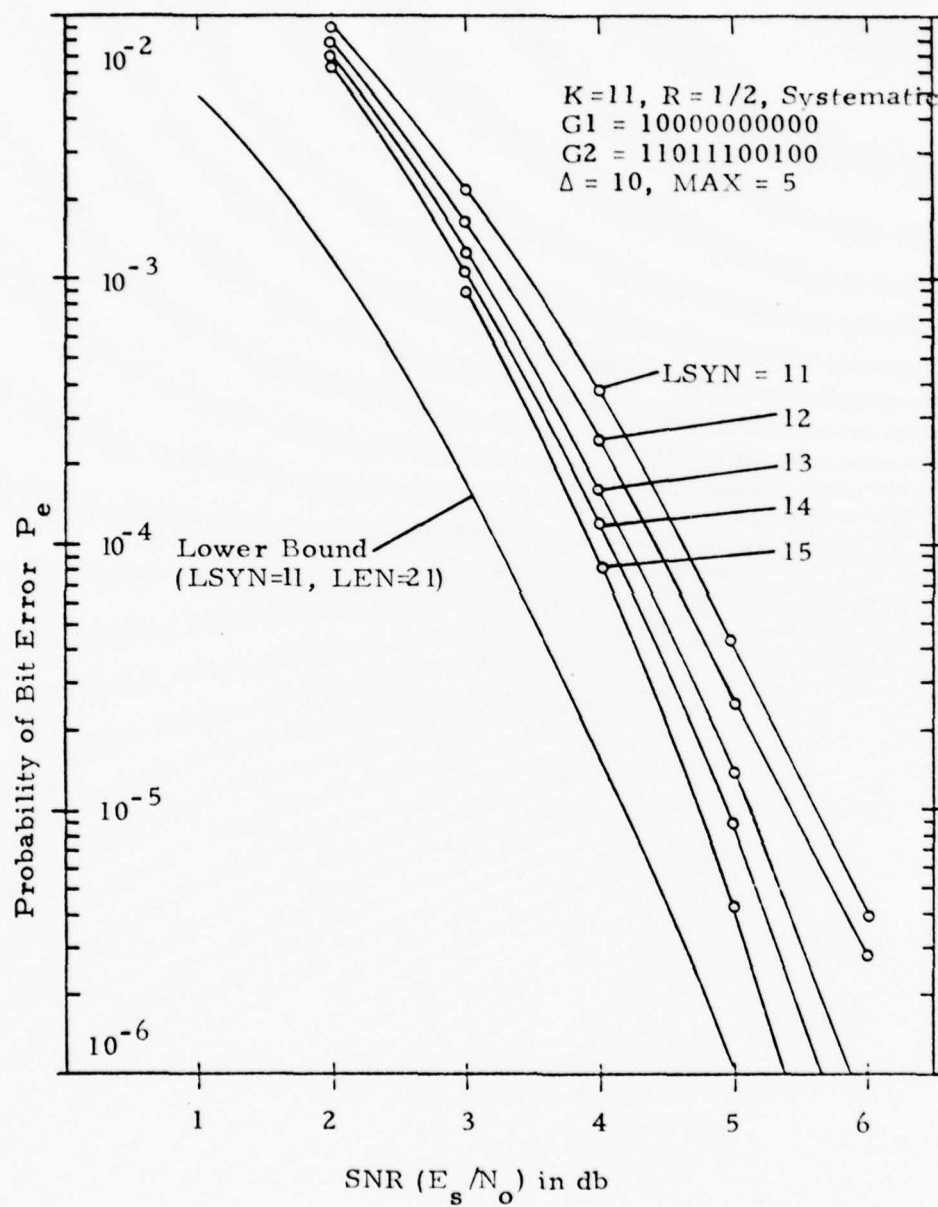
Figure 5.8 SUBMAP Predicted Performance, K = 11

Also plotted in Figures 5.6 and 5.7 are extracted Viterbi

decoding results as published by Heller and Jacobs [17] for K = 5

and K =7 respectively. The side-by-side comparison of Viterbi

decoding with SUBMAP decoding indicates the greater range of

coding gain achievable by the more complex Viterbi approach. This

point will be touched on again in Section 5.4 where a loose lower

bound on SUBMAP performance (for a fixed decoding delay $\Delta$) will be

seen to lie above the published Viterbi performance.

5.4 A Loose Lower Bound on Probability of Error

The previous sections have shown that for a given code and

delay $\Delta$ one can achieve improved SUBMAP performance by using

longer syndrome vectors as shown in Figures 5.6 through 5.8.

It would be useful, when comparing the suboptimal SUBMAP decoder

performance with other decoder approaches to have some lower

bound on SUBMAP probability of error. This would enable one to

state a performance improvement limit which even continued

effort at increasing the SUBMAP look-up table size would not

achieve. A computational method of specifying such a lower bound

for a given delay and code can be obtained by examining the

syndrome equation set. If one hypothesizes that no prior incorrect

error estimates have been made and that all prior estimates have

been fedback, one can examine the resultant first error probability

to obtain a bound.

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As an illustration, consider the previously discussed K=5, rate $\frac{1}{2}$ systematic code for which the first six syndrome equations are:

$$Z_{t-5} = e_{t-5}^{(2)} + e_{t-5}^{(1)} + e_{t-6}^{(1)} + e_{t-7}^{(1)} + 0 + e_{t-9}^{(1)}$$

$$Z_{t-4} = e_{t-4}^{(2)} + e_{t-4}^{(1)} + e_{t-5}^{(1)} + e_{t-6}^{(1)} + 0 + e_{t-8}^{(1)}$$

$$Z_{t-3} = e_{t-3}^{(2)} + e_{t-3}^{(1)} + e_{t-4}^{(1)} + e_{t-5}^{(1)} + 0 + e_{t-7}^{(1)}$$

$$Z_{t-2} = e_{t-2}^{(2)} + e_{t-2}^{(1)} + e_{t-3}^{(1)} + e_{t-4}^{(1)} + 0 + e_{t-6}^{(1)}$$

$$Z_{t-1} = e_{t-1}^{(2)} + e_{t-1}^{(1)} + e_{t-2}^{(1)} + e_{t-3}^{(1)} + 0 + e_{t-5}^{(1)}$$

$$Z_{t} = e_{t}^{(2)} + e_{t}^{(1)} + e_{t-1}^{(1)} + e_{t-2}^{(1)} + 0 + e_{t-4}^{(1)} \qquad (5.23)$$

Here with $\Delta = 5$ the decoder is now to estimate $e_{t-5}^{(1)}$.

If one now considers the conditional probability of the decoder making a mistake in estimating $e_{t-5}^{(1)}$ conditioned on all prior estimates being both correct and correctly fedback, the following set of modified syndrome equations apply.

$$Z'_{t-5} = e_{t-5}^{(2)} + e_{t-5}^{(1)}$$

$$Z'_{t-4} = e_{t-4}^{(2)} + e_{t-4}^{(1)} + e_{t-5}^{(1)}$$

$$Z'_{t-3} = e_{t-3}^{(2)} + e_{t-3}^{(1)} + e_{t-4}^{(1)} + e_{t-5}^{(1)}$$

$$Z'_{t-2} = e_{t-2}^{(2)} + e_{t-2}^{(1)} + e_{t-3}^{(1)} + e_{t-4}^{(1)} + 0$$

$$Z_{t-1} = e_{t-1}^{(2)} + e_{t-1}^{(1)} + e_{t-2}^{(1)} + e_{t-3}^{(1)} + 0 + e_{t-5}^{(1)}$$

$$Z_{t} = e_{t}^{(2)} + e_{t}^{(1)} + e_{t-1}^{(1)} + e_{t-2}^{(1)} + 0 + e_{t-4}^{(1)} \qquad (5.24)$$

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Here the upper right triangle of error bits in (5.23) has been removed and the associated syndrome bits modified to corrected values to reflect the error bit removal through feedback.

Faced with the modified set of syndrome equations for this idealized conditional case, one can compute a ROMMAP or SUBMAP look-up table as previously discussed. Then the conditional probability of incorrectly estimating $e_{t-5}^{(1)}$ can be computed exactly in the ROMMAP case or bounded in the SUBMAP case as explained in Sections 5.1.3 and 5.2.2 respectively. In the ROMMAP case this idealized probability of error will be a lower bound on ROMMAP *performance no matter how many additional syndrome bits* might be used. In the SUBMAP case the idealized probability will be a tight upper bound on the $P_e$ and being a tight upper bound for such an idealized case will be useful as a loose lower bound to actual performance.

This SUBMAP lower bound has been computed for the K=5 ($\Delta$=5) and K=11 ($\Delta$=10) decoders and also plotted on Figure 5.6 and 5.8 respectively. Since the assumption of no previous incorrect error estimates is quite idealized, this first error probability lower bound is relatively loose. Indeed as shown in later simulation results for the K=5 code in Figure 5.9, continued increases in syndrome length with a fixed delay causes performance to tend

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toward an improvement limit ≤1 db above the loose lower bound.

## 5.5 Simulation Results

In this section performance curves from Monte Carlo simulation for a number of SUBMAP decoder implementations are presented. Following some preliminary explanation regarding simulation methods, actual SUBMAP performance results for two system atic codes listed by Bussgang [19] for constraint lengths of K=5 and K=11 are given. Although primary emphasis is placed on SUBMAP decoder performance, some cases were simulated in which the conventional feedback decoding approach (using minimum weight error vectors) as well as the SUBMAP decoder were run with the same error inputs to enable a side-by-side comparison of the two methods. The performance curves show the improvement possible not only by using longer constraint length codes but also that by using more syndrome bits in the SUBMAP decoding of a particular code. Also displayed is the extent of performance improvement achievable with no complexity increase by merely substituting the SUBMAP table for the conventional feedback look-up tables.

## 5.5.1 Method of Simulation

The simulation described in the following section was done in FORTRAN on a Digital Equipment Corporation KL 10 time-sharing

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system (in batch mode) located at the University of Southern California. All codes considered were optimum rate $\frac{1}{2}$ systematic codes given in Bussgang [19]. Two uniform pseudo-random number generators (PRN) were used interchangeably at critical points to check the quality of the error generation process. Appendix B gives additional details on these PRN generators. To insure the statistical validity of simulation results, error bit inputs to the two syndrome former input ports were:

(1) independently generated through separate calls to the PRN generator using seeds selected to guarantee no repetition during $10 \times 10^6$ calls to the PRN generator; and

(2) simulated for at least $10^6$ error bits per syndrome former input except at higher SNR levels were up to $5 \times 10^6$ error bits per input were simulated. All $P_e$ points plotted herein are based on at least 100 decoding errors.

The general operation of the simulation program consisted of reading in and storing the appropriate look-up table previously generated in accordance with Section 5.2. This look-up table would have been created as either a SUBMAP table or as a minimum weight table corresponding to the decoder type being simulated. The PRN generator then supplied two PRNs uniformly distributed between 0 and 1. These were used to create $e_i^{(1)}$ and $e_i^{(2)}$ error bit inputs

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based on the BSC crossover probability given by the symbol SNR $(E_s/N_0)$ as specified in Section 5.1.2. The syndrome former then computed the syndrome bit $Z_i$ for input to the syndrome vector while error bits were shifted along each clock period in an $\underline{E}$ vector. These two error bits were actually input to two parallel decoders, one decoding based strictly on the current $\underline{Z}$ with no feedback of error estimates and a second using feedback of error decisions to update the syndrome vector. Thus actual performance for a particular look-up table could be simulated in one pass both with and without feedback. The actual $P_e$ for both decoders was computed by counting the number of times during the simulation that the estimated error $\hat{e}_{i-\Delta}^{(1)}$ did not equal the actual error bit $e_{i-\Delta}^{(1)}$ and dividing by the number of error bits processed during the simulation through one input to the syndrome former.

Since the operation of the syndrome former is independent of the actual data that might have been input to the encoder, the above simulation effectively assumes all zero data and thus only error inputs entering the syndrome former.

The PRN generators used were such that, given a starting "seed" number, the generator would output the identical stream of PRNs. Thus it was possible to run simulations of various SUBMAP and conventional feedback schemes using the identical input error bit streams. This gives a more reasonable comparison of techniques

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than non-identical input streams.

### 5.5.2 Decoder Performance for K=5 Code

The decoding of the K=5 binary systematic rate $\frac{1}{2}$ code (G1 = 10000, G2 = 11101) used in previous performance prediction discussions was simulated for both the SUBMAP and the conventional feedback approaches. As described in Section 5.5.1, the performance of each decoder table was simulated both with and without feedback. The decoding delay was fixed at $\Delta = 5$ (i.e. estimate $e_{i-5}^{(1)}$) with MAX = 5 in the table creation step. The design SNR was always 2 db ($E_s/N_0$).

Figure 5.9 shows simulation results for the K=5 code SUBMAP decoder with and without feedback for LSYN = 6 and LSYN = 11. Also simulated but not shown was LSYN = 15. Figure 5.9 also gives computed upper bounds and the loose lower bound for $\Delta = 5$. It can be seen that the simulated performance falls extremely close to the predicted tight upper bounds and well above the loose lower bound. The futility of increasing LSYN much beyond two constraint lengths is actually indicated by what is not plotted since both predicted performance and simulation results for LSYN = 15 fell so close to LSYN = 11 bound and results so as to make plotting them infeasible.

Also evident is the minimal improvement achievable by using feedback with the SUBMAP decoder. Indeed using feedback in the LSYN = 11 case produced somewhat poorer simulated performance.
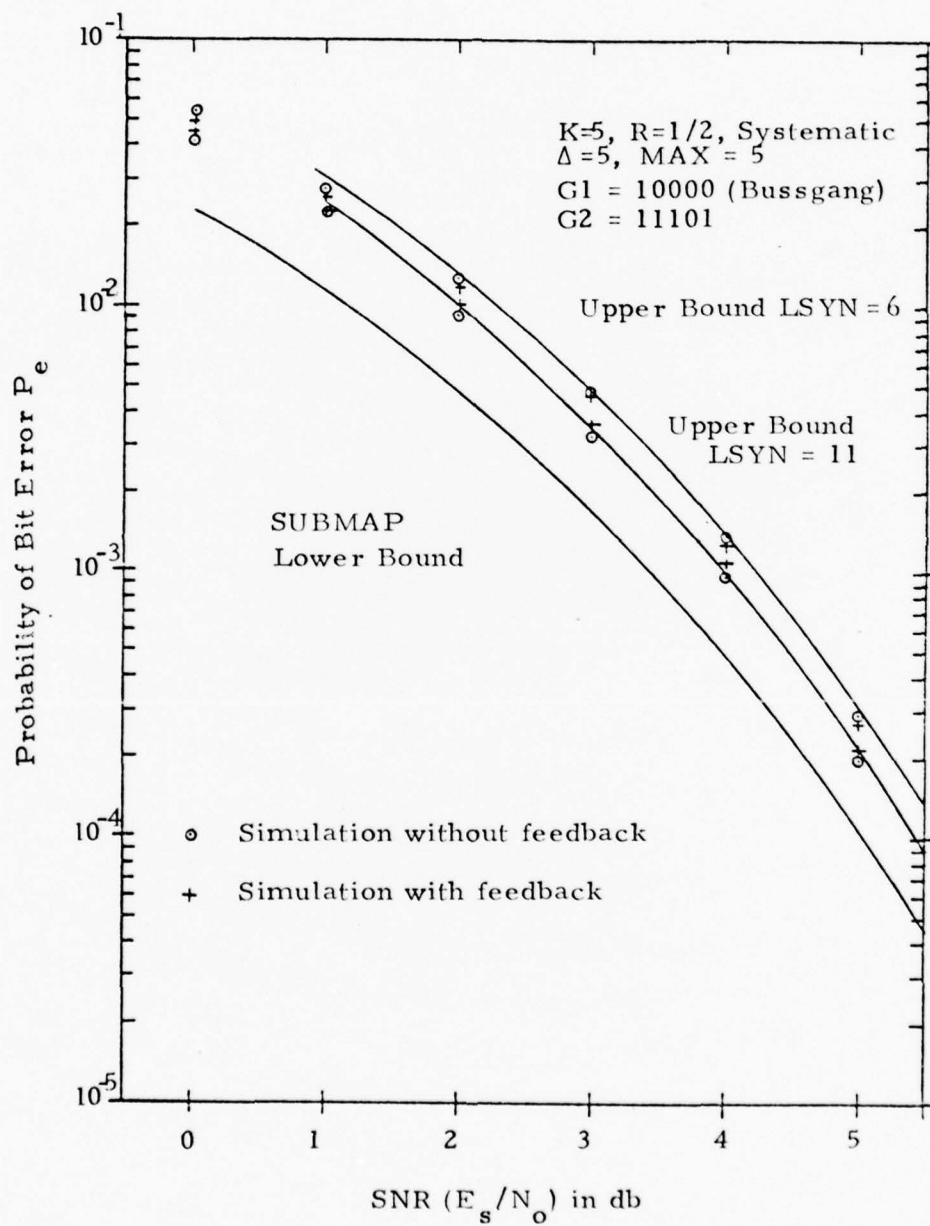
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Figure 5.9 SUBMAP Simulation, K = 5
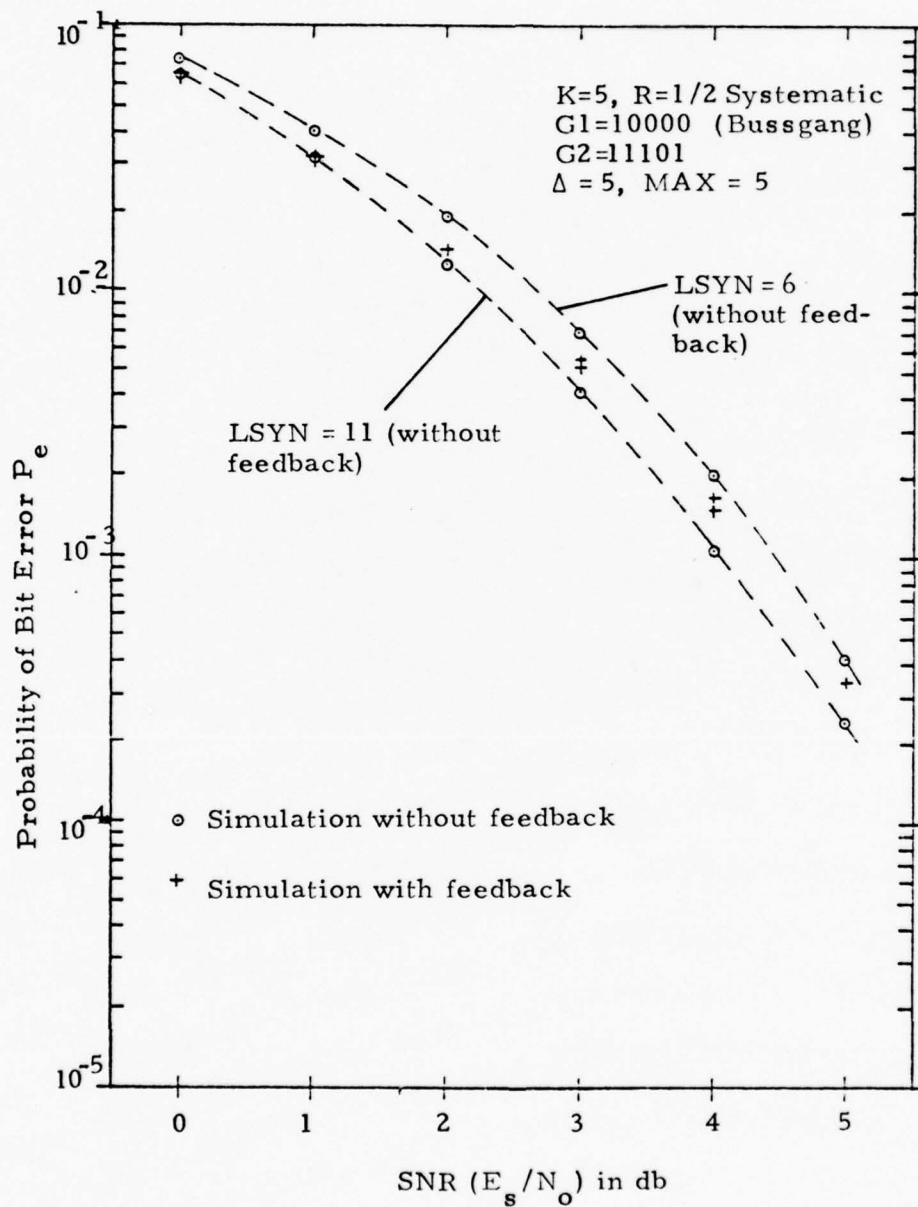
Figure 5.10 Minimum Weight Simulation, K=5

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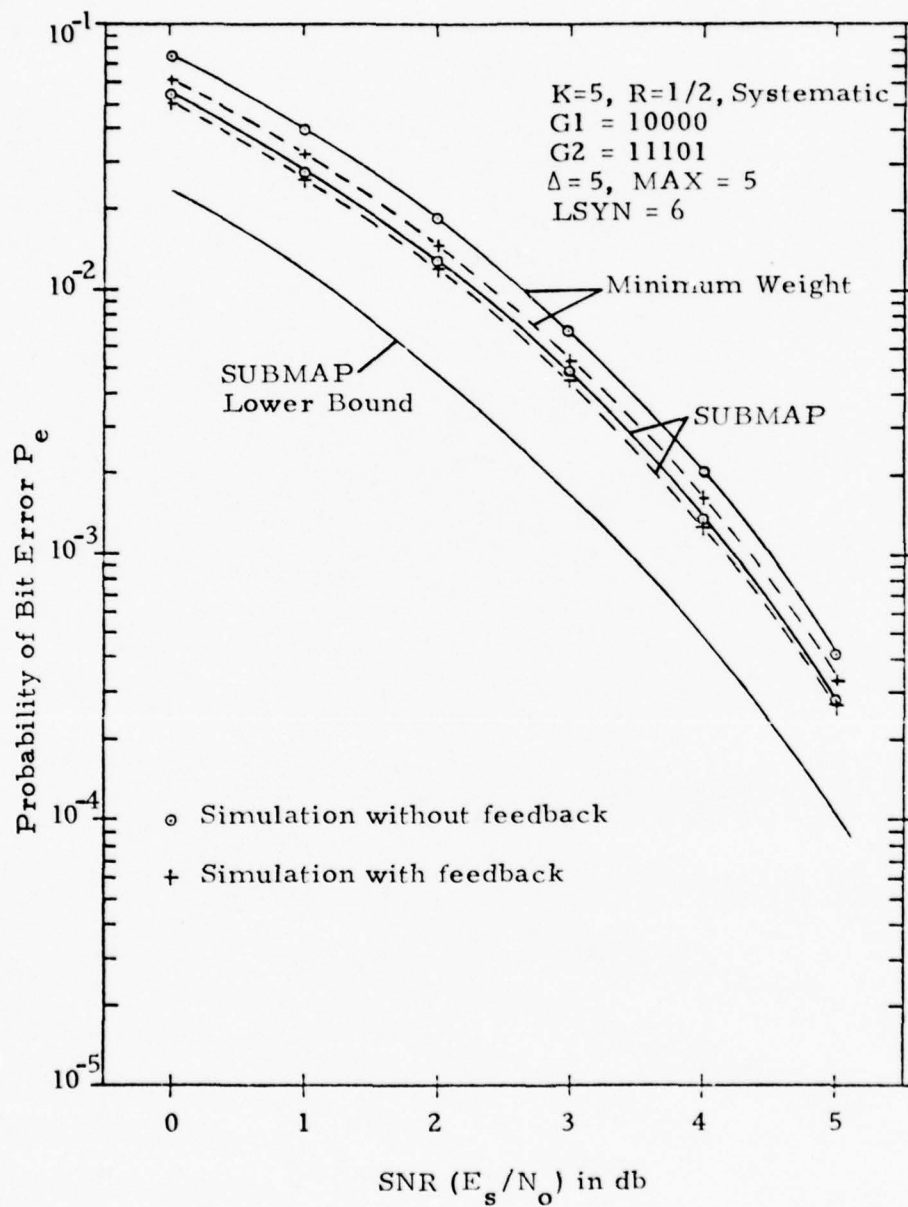
Figure 5.11 Comparison of SUBMAP and Conventional
Feedback, K = 5

Figure 5.10 shows simulation results for a minimum weight look-up table decoder with and without feedback for the same K=5 code for LSYN=6 and LSYN=11. The improvement due to feedback is seen to be more substantial for the minimum weight case than for the SUBMAP case at LSYN=6. However at LSYN = 11 feedback appears to be counterproductive since feedback results for the minimum weight approach practically coincide for both LSYN = 6 and LSYN =11.

Figure 5.11 combines selected SUBMAP and conventional feedback results on one graph to show the relative superiority of the SUBMAP approach over the minimum weight approach. A careful comparison indicates that the SUBMAP approach even without feedback slightly outperforms conventional feedback decoding. However the absolute performance improvement for the short (K=5) constraint length code and LSYN combinations considered was roughly 0.2 dB at lower SNR levels.

In summary, although these simulation results confirm the accuracy of the predicted SUBMAP performance bounds, they reveal only a minimal improvement over conventional feedback decoder performance. Therefore the following section deals with SUBMAP performance for a longer constraint length (K=11) code.

5.5.3 Decoder Performance for K=11 Code

The SUBMAP decoding of the K=11 systematic rate $\frac{1}{2}$ code specified by

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$$G1 = 10000000000$$

$$(5.25)$$

$$G2 = 11011100100$$

was simulated both with and without feedback for LSYN = 11 and

LSYN = 13. A design SNR of 2 db $(E_s/N_0)$ was used with MAX=5

for look-up table creation. The decoding delay was fixed at $\Delta = 10$

(i.e. estimate $e_{i=10}^{(1)}$). Also simulated was decoding using a mini-

mum weight approach as used in conventional feedback.

Figure 5.12 shows SUBMAP simulation results for the K=11

code for LSYN = 11 and LSYN = 13. Evident is a slight perform-

ance improvement obtained by using feedback with the SUBMAP

table at both LSYN values. Shown as solid lines are the predicted

upper bounds and the loose lower bound. Again the accuracy of

the upper bounds is confirmed.

Figure 5.13 shows simulation results using a minimum weight

look-up table to demonstrate conventional feedback performance for

the K = 11 code at LSYN = 11 and 13. Shown as solid lines are the

predicted upper bounds and the loose lower bound. Evident is the

performance improvement obtained by using feedback.

Figure 5.14 combines SUBMAP and conventional feedback

results for LSYN = 11 on one graph to show the relative superiority

of the SUBMAP approach. Thus is evident the performance

improvement possible with no decoding complexity increase by

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merely substituting a SUBMAP look-up table for the minimum
weight table in a conventional feedback decoder.

The statement was made in Section 5.3 that it is better to use
a longer constraint length code rather than merely increasing LSYN
for a shorter constraint length code. This fact is shown to be true
in Figure 5.15 where simulation results for K=5 (LSYN = 11) and
K = 11 (LSYN =11) have been compared. It is evident that the K=11
code is preferable if hardware complexity equivalent to a size
$2^{LSYN} = 2^{11}$ is feasible. The improvement is to be expected since
the minimum distance of the K=5 code was 4 while that of the K=11
code was 7. The additional error correction capability exhibits
itself in the steeper slope to the K=11 curve of $P_e$.
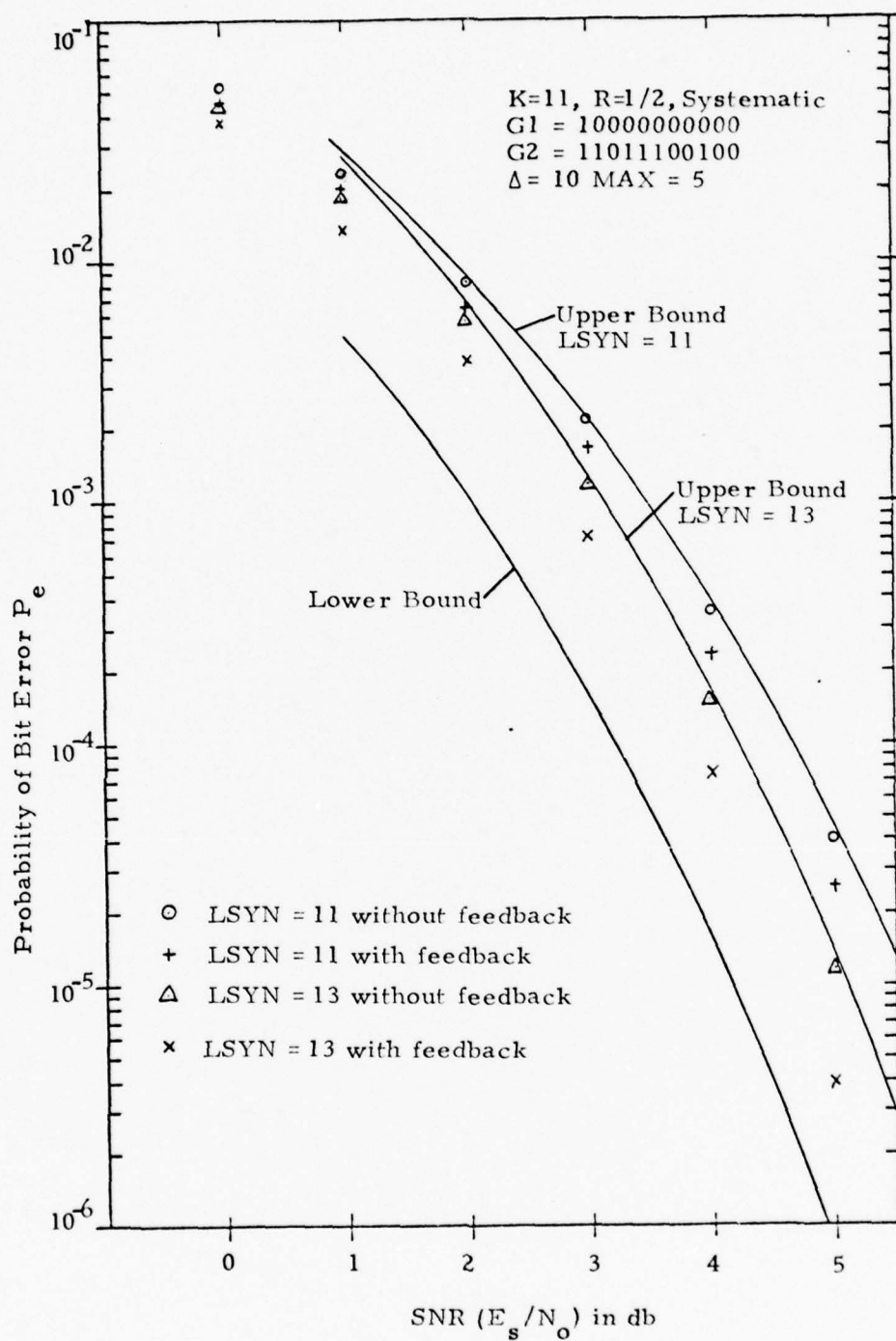
Figure 5.12  SUBMAP Simulation, K=11

The figure shows a plot with:
- Y-axis: Probability of Bit Error $P_e$ (ranging from $10^{-6}$ to $10^{-1}$)
- X-axis: SNR ($E_s/N_o$) in db (ranging from 0 to 5)

Plot annotations:
- K=11, R=1/2, Systematic
- G1 = 10000000000
- G2 = 11011100100
- $\Delta$ = 10 MAX = 5
- Upper Bound LSYN = 11
- Upper Bound LSYN = 13
- Lower Bound

Legend:
- ○  LSYN = 11 without feedback
- +  LSYN = 11 with feedback
- △  LSYN = 13 without feedback
- ×  LSYN = 13 with feedback

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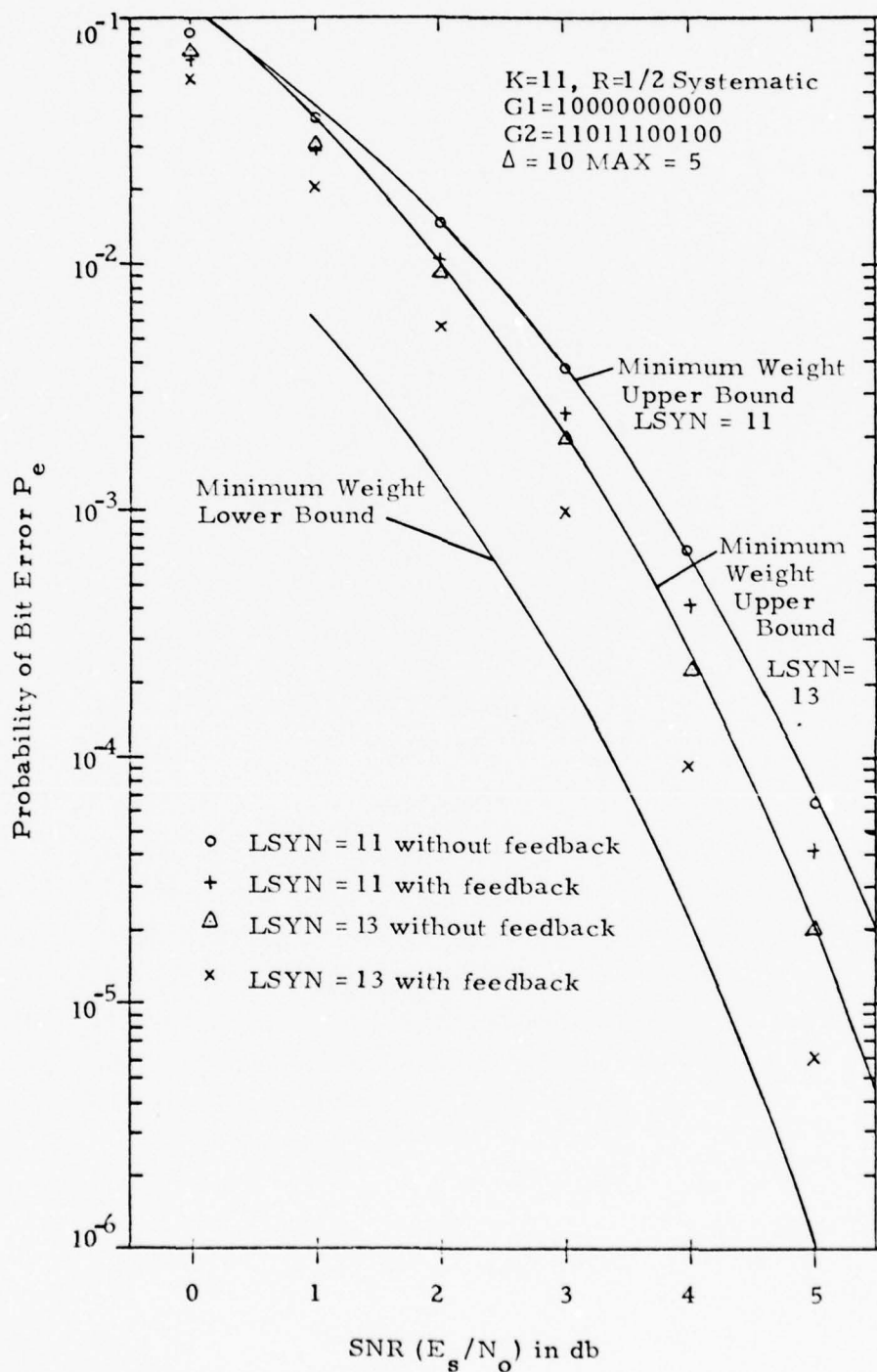
Figure 5.13 Minimum Weight Simulation, K = 11          152
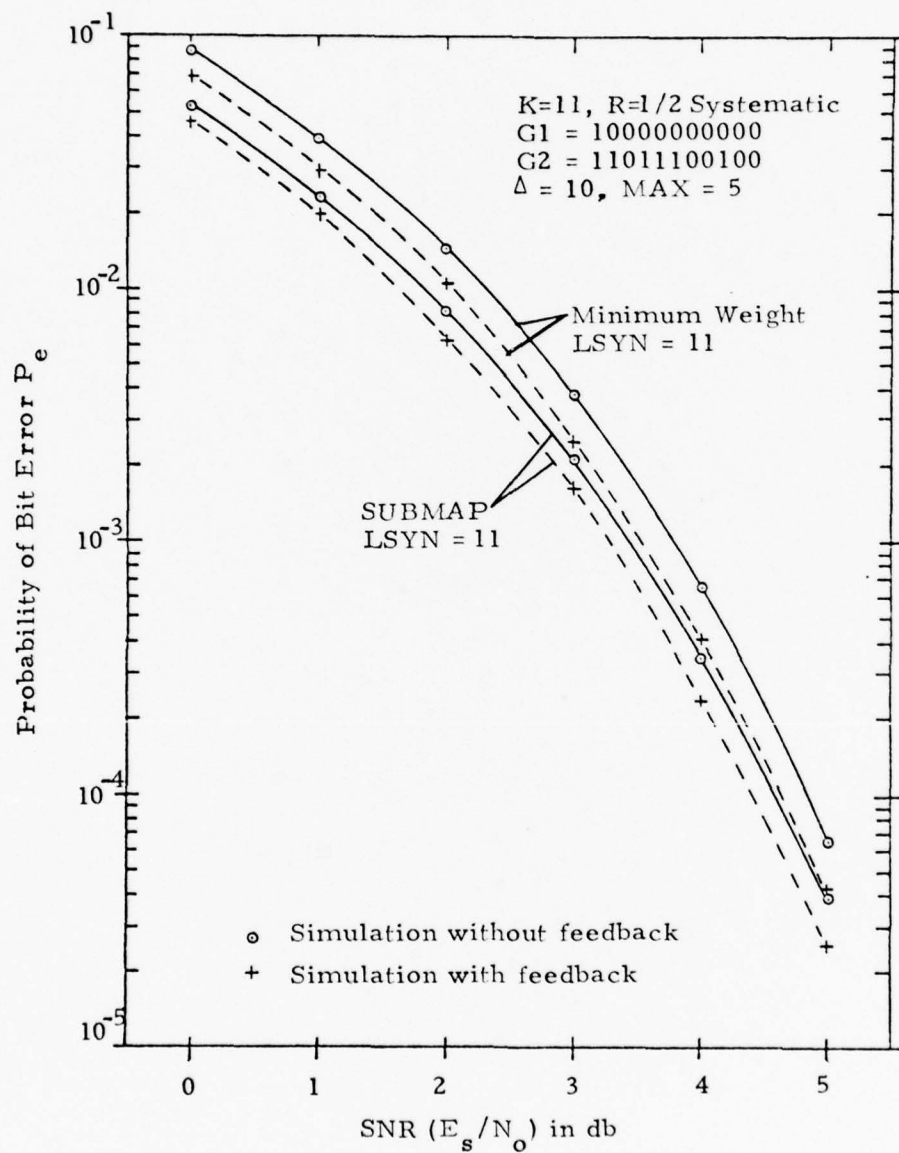
Figure 5.14 Comparison of SUBMAP and Conventional
Feedback, K = 11

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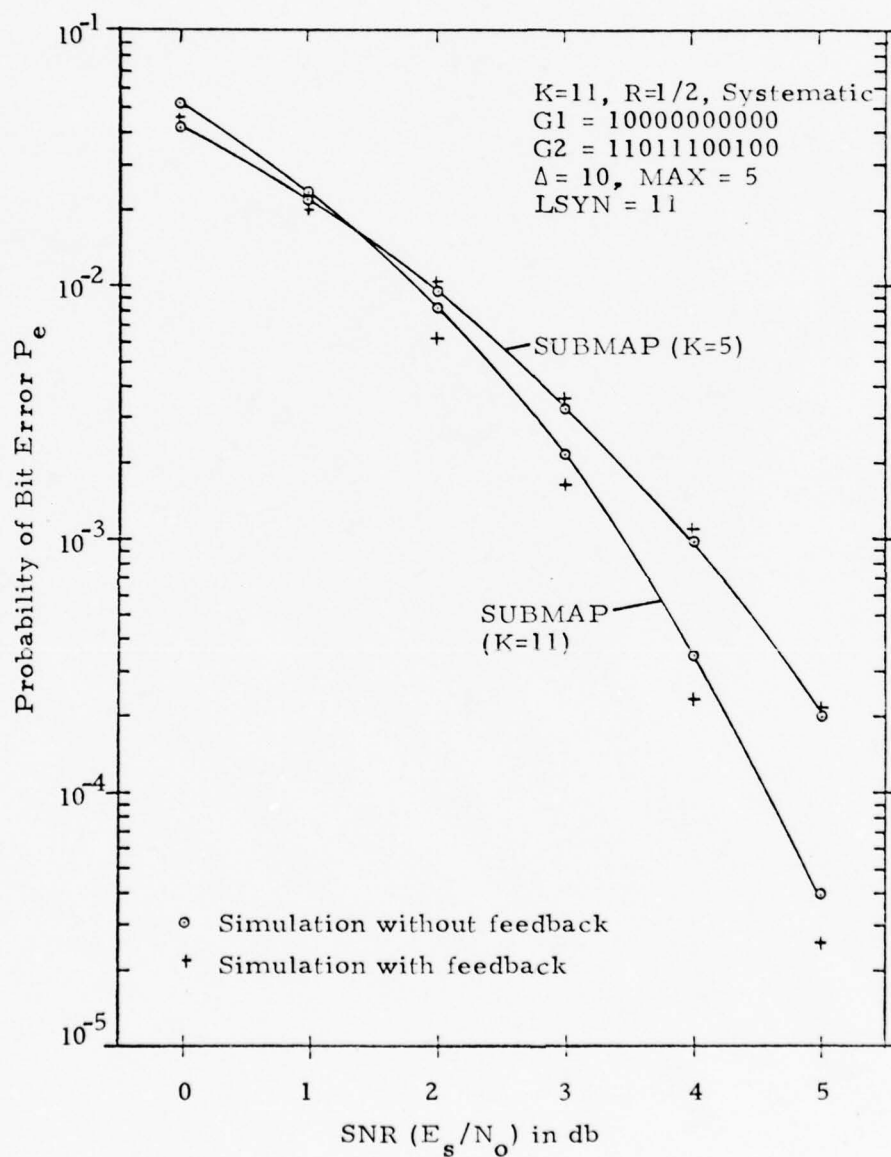
Figure 5.15 Effect of Constraint Length on SUBMAP
Performance

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Chapter 6

CONCLUSIONS AND RECOMMENDATIONS

This investigation developed in Chapter 3 several optimal MAP error bit estimation algorithms for decoding convolutional codes transmitted over a binary symmetric channel. These represent a set of error decoding approaches complementary to similar techniques previously used for data estimation. One advantage of such error bit estimation using syndromes is that the resulting performance is independent of a priori data probability. Thus an arbitrary actual data source output can be communicated reliably without having to preprocess the data prior to encoding just to insure equilikely input bits. This is in contrast to standard Viterbi maximum likelihood sequence decoding which requires equilikely input probabilities. Also in Chapter 3 corresponding optimal MAP error algorithms have been derived to handle the more realistic case of error decoding under a real time or finite delay restriction.

During the Chapter 4 investigation of implementable ROMMAP and SUBMAP suboptimal decoders using MAP error bit estimation with finite delay, it was seen that conventional feedback decoding is a rough approximation to MAP error bit estimation. The results of simulation discussed in Chapter 5 showed that a SUBMAP decoder based on MAP estimation outperforms conventional feedback decoding

for the same complexity. Noted also was the fact that feeding back error decisions in the SUBMAP case does provide additional improvement.

The complexity parameter for ROMMAP, SUBMAP, and conventional feedback is basically the look-up table size ($2^{LSYN}$). Thus, based on the simulation results of Chapter 5, for a given complexity one is inclined to favor use of a SUBMAP look-up table instead of a minimum weight look-up table as used in conventional feedback decoders. One other point worth mentioning in this connection is the comparative "weights" of the look-up tables, that is the number of 1's in the table entries. In all cases considered here, the SUBMAP table always had significantly less 1's than did the minimum weight table of the same $2^{LSYN}$ size. For example Table 6.1 shows comparative weights for the K = 11 systematic code at LSYN = 11 and 13. This investigation assumed and simulated read only memory (ROM) look-up tables in which the syndrome vector $\underline{Z}$ was converted to an address in a $2^{LSYN}$-size table where each address location contains either a 1 or 0. However if an alternative device such as one based on an associative memory were used, the SUBMAP table implementation would be much simpler than that of the minimum weight table. This is due to the smaller number of 1-associated syndromes $\underline{Z}$ (i.e. the lighter-weight table) of the SUBMAP approach which would require a significantly smaller associative memory

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| LSYN | Look Up Table Size | Table Type | Design SNR ($E_s/N_0$ in db | | | | | | |
|------|------|------|------|------|------|------|------|------|------|
| | | | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 11 | 2048 | SUBMAP Table | 117 | 144 | 149 | 151 | 152 | 152 | 152 |
| | | Min. Wt. Table | 618 | 618 | 618 | 618 | 618 | 618 | 618 |
| 13 | 8192 | SUBMAP Table | 562 | 615 | 676 | 724 | 724 | 724 | 724 |
| | | Min. Wt. Table | 2300 | 2300 | 2300 | 2300 | 2300 | 2300 | 2300 |

Table 6.1.  Look-up Table Weights for K = 11 Code Example.

device.  This appears to be an area worthy of some additional investi-

gation in light of continuing advances in LSI capability.

The simulation results of Chapter 5 also demonstrated the accu-

racy of the tight upper bounds on performance derived in Section 5.2.2

thereby implying that these bounds represent a valuable design tool

for predicting actual decoder performance.  Section 5.3 confirmed

the expected improvement in SUBMAP decoder performance with

increasing syndrome vector length.  This section also validated the

rule of thumb that, with a complexity limit on syndrome vector length,

the SUBMAP designer should select a code having as long a constraint

length as possible within this syndrome length for best SUBMAP

performance.

The practical hardware limits to increasing look-up table size is obviously a major limitation on continued SUBMAP performance improvement. Thus it is felt that further development of the techniques derived in Chapter 3 for optimal error bit estimation could lead to an implementation scheme that would bridge the gap between the performance of the look-up table approach of SUBMAP(and indeed of conventional feedback) and that of Viterbi maximum likelihood sequence decoding. Bridging this gap with such a MAP error bit estimation approach based on syndromes would enable decoder performance to be improved yet still maintain performance independent of a priori input data probability.

# APPENDIX A

## VITERBI DECODING WITH

## NONEQUILIKELY INPUT SEQUENCES

As stated by Viterbi [14],

> ...for completely general channels... if all input data
> are equally likely, the decoder which minimizes the
> error probability is one which compares the condition-
> al probabilities, also called likelihood functions,
> $P(\underline{y}/\underline{x}^{(\bullet)})$ where $\underline{y}$ is the overall received sequence and
> $\underline{x}^{\bullet}$ is one of the possible transmitted sequences, and
> decides in favor of the maximum. This is called a
> maximum likelihood decoder. The likelihood functions
> are given or computed from the specifications of the
> channel. Generally it is more convenient to compare
> the quantities $\log P(\underline{y}/\underline{x}^{(\bullet)})$, called log likelihood func-
> tions and the result is unaltered since the logarithm is
> a monotonic function of its (always positive) argument.

The important point for the reader to bear in mind is that Viterbi's

excellent description of his now famous algorithm applies only to the

case of equally likely input sequences.

To analyze the case where the input data sequences are not equi-

likely, a maximum a posteriori decoder should be considered. This

is one which compares the conditional probabilities

$$P(\underline{x}^{(m)}/\underline{y}) = \frac{P(\underline{y}/\underline{x}^{(m)})P(\underline{x}^{(m)})}{P(\underline{y})} \tag{A.1}$$

for candidate sequences $\{\underline{x}^{(m)}\}$ and decides in favor of the maximum.

Now consider the specific case of Section 3.2 where a rate 1/3

decoding problem has been converted to a rate 1/2 encoder and

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channel with error bit sequence inputs. This provides a good example of what adaptation is necessary for a Viterbi algorithm for a nonequilikely input case. Thus Figure 3.3.b is the "error encoder" that will now be analyzed. Here define $P(e_t^{(i)} = 1) = p$ and $P(e_t^{(i)} = 0) = (1-p)$ since the binary symmetric channel still applies. In this case Eq. A.1 becomes

$$\underset{\text{over all } \underline{c}(\underline{E})}{\text{maximize}} \left\{ P(\underline{c}(\underline{E})/\underline{Z}) = \frac{P(\underline{Z}/\underline{c}(\underline{E})P(\underline{c}(\underline{E}))}{P(\underline{Z})} \right\} \qquad (A.2)$$

where to obtain A.2 the following substitutions have been made:

(1) $\underline{y}$ becomes $\underline{Z}$ = the vector of received syndrome bits;

(2) $\underline{x}^{(m)}$ becomes $\underline{c}(\underline{E})$ = one of the possible "transmitted" sequences; and

(3) $\underline{E} \overset{\Delta}{=} (e_0, e_1, \ldots, e_N)$ = input error vector.

Note that each possible input error vector $\underline{E}$ corresponds to only one $\underline{c}(\underline{E})$ just as a data encoder output maps directly back to one data input sequence in the more conventional data encoding. Due to the bit-to-bit independence of all error bits, A.2 can be rewritten as:

$$\underset{\text{all } \underline{c}(\underline{E})}{\text{maximize over}} \left\{ P(\underline{c}(\underline{E})/\underline{Z}) = \prod_{i=0}^{N-1} P(Z_i/c_i(\underline{E}))P(c_i(\underline{E})) \right\} . \qquad (A.3)$$

The elements in Eq. A.3 are:

(1) $Z_i = \begin{bmatrix} Z_i^{(2)} \\ \\ Z_i^{(3)} \end{bmatrix}$ = "received" syndrome bits at time i;

(2) $c_i(\underline{E}) = \begin{bmatrix} c_i^{(2)} \\ \\ c_i^{(3)} \end{bmatrix}$ = "transmitted" bits at time i; and

(3) N = length of the input error sequence.

Taking logarithms yields:

$$\text{maximize} \atop \text{over all } \underline{c}(\underline{E}) \left\{ \log P(\underline{c}(\underline{E})/\underline{Z}) = \sum_{i=0}^{N-1} \log(P(Z_i/c_i(\underline{E}))P(c_i(\underline{E}))) \right\} \quad \text{(A.4)}$$

where the sum is over all branches in the error encoder trellis path corresponding to this particular $\underline{E}$ vector.

Now define $H(\underline{a}, \underline{b}) \triangleq$ Hamming distance between vectors $\underline{a}$ and $\underline{b}$. Also note that the $P(c_i(\underline{E}))$ is just the a priori probability of the input error bit $e_i^{(1)}$ for the $\underline{c}(\underline{E})$ in question (either p or 1-p since the "inputs" are characterized by the binary symmetric channel crossover probability and there is a one-to-one mapping from inputs to outputs). Hence,

$$\text{maximize} \atop \text{over all } \underline{c}(\underline{E}) \left\{ \log P(\underline{c}(\underline{E})/\underline{Z}) \right.$$

$$\left. = \sum_{i=0}^{N-1} \log(p^{H(Z_i, c_i(\underline{E}))}(1-p)^{n_0 - H(Z_i, c_i(\underline{E}))} P(e_i^{(1)})) \right\} \quad \text{(A.5)}$$

$$= \text{maximize} \atop \text{over all } \underline{c}(\underline{E}) \left\{ \sum_{i=0}^{N-1} [H(Z_i, c_i(\underline{E}))\log p + (n_0 - H(Z_i, c(\underline{E}))\log(1-p) + \log P(e_i^{(1)})] \right\}$$

$$\text{(A.5)}$$

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where $n_0$ = the number of bits in $Z_i$ and $c_i(\underline{E})$ ($n_0 = 2$ in this specific case).

Rearranging yields:

$$\begin{array}{l} \text{maximize} \\ \text{over all } \underline{c}(\underline{E}) \end{array} \left\{ \sum_{i=0}^{N-1} [-H(Z_i, c_i(\underline{E})) \log \frac{1-p}{p} + \overbrace{n_0 \log(1-p)}^{\text{constant}} \right. \\ \left. \qquad\qquad + \log P(e_i)] \right\} \,. \qquad\qquad (A.6)$$

It should be noted that the third term of the summation will equal $\log(p)$ or $\log(1-p)$ depending on the value of $e_i^{(1)}$ specified by the error input vector $\underline{E}$ being considered. Also, if $P(e_i^{(1)} = 1) = P(e_i^{(1)} = 0) = \frac{1}{2}$ (as would be true <u>only</u> if the error inputs were equilikely), Eq. A.6 would reduce to the conventional Viterbi decoding rule:

$$\begin{array}{l} \text{maximize} \\ \text{over all } \underline{c}(\underline{E}) \end{array} \left\{ \sum_{i=0}^{N-1} [-H(Z_i, c_i(\underline{E}))K_1 + K_2] \right\} \qquad\qquad (A.7)$$

with $K_1 = \log\left(\frac{1-p}{p}\right)$ and $K_2 = n_0 \log(1-p) + \log(\frac{1}{2})$.

This would direct the decoder to select the path $\underline{c}(\underline{E})$ minimizing the Hamming distance between $\underline{Z}$ and $\underline{c}(\underline{E})$.

On the contrary, for the nonequilikely input case under consideration here, $P(e_i^{(1)} = 1) \neq P(e_i^{(1)} = 0)$. Thus $\log P(e_i^{(1)})$ must be included in all computations and a modified metric must be used. Generally A.6 requires that for a maximum, the trellis path would be selected which both:

(1) minimizes $H(\underline{Z}, \underline{c}(\underline{E}))$ weighted by $\log\left(\frac{1-p}{p}\right)$; and

(2) maximizes $\log P(\underline{E}) = \sum_{i=0}^{N-1} \log P(e_i^{(1)})$.

The quantity $\log P(e_i^{(1)})$ is either $\log(p)$ or $\log(1-p)$. Since $0 \leq p \leq 1$ and $0 \leq 1-p \leq 1$, this logarithm is nonpositive. The least negative logarithm corresponds to $e_i^{(1)} = 0$. Thus the maximization in A.6 implies that in selecting $\widehat{\underline{E}}$ one should "lean" toward an input sequence with minimum weight whose output sequence $\underline{c}(\widehat{\underline{E}})$ is closest to the actual received syndrome sequence $\underline{Z}$.

With the definition $H(\underline{E}, 0) \overset{\Delta}{=}$ Hamming distance between a candidate $\underline{E}$ and the all-zeros vector, A.6 can be expressed entirely in terms of Hamming distances. Neglecting constant terms, the A.6 rule can be written in an equivalent form as:

$$\underset{\text{over all } \underline{c}(\underline{E})}{\text{maximize}} \left\{ \log\left( \frac{p^{H(\underline{Z}, \underline{c}(\underline{E}))} p^{H(\underline{E}, \underline{0})}}{(1-p)^{H(\underline{Z}, \underline{c}(\underline{E}))}(1-p)^{H(\underline{E}, \underline{0})}} \right) \right\} \tag{A.7}$$

$$\Rightarrow \underset{\text{over all } \underline{c}(\underline{E})}{\text{maximize}} \left\{ \log\left[ \left(\frac{1-p}{p}\right)^{-[H(\underline{Z}, \underline{c}(\underline{E})) + H(\underline{E}, \underline{0})]} \right] \right\}$$

$$\Rightarrow \underset{\text{over all } \underline{c}(\underline{E})}{\text{maximize}} \left\{ -[H(Z, c(E)) + H(E, 0)] \underbrace{\log\left(\frac{1-p}{p}\right)}_{\text{constant}} \right\}$$

$$\Rightarrow \underset{\text{over all } \underline{c}(\underline{E})}{\text{minimize}} [H(\underline{Z}, \underline{c}(\underline{E})) + H(\underline{E}, \underline{0})] \quad \begin{array}{l} \text{Optimum decoding} \\ \text{rule for nonequilikely} \\ \text{inputs.} \end{array} \tag{A.8}$$

Thus the decoding rule for error sequences in this case requires the selection of $\widehat{\underline{E}}$ as that input error sequence which minimizes the Hamming distance <u>sum</u> given by A. 8.

Since the Hamming distance $H(\underline{E}, \underline{0})$ is the same as the Hamming weight of the error vector $\underline{E}$, the decoding rule for nonequilikely input sequences can also be written as:

$$\boxed{\underset{\text{over all } \underline{c}(\underline{E})}{\text{minimize}} \left\{ [H(\underline{Z}, \underline{c}(\underline{E})) + W(\underline{E})] \right\}} \qquad (A.9)$$

where $W(\underline{E})$ is the weight of the error vector $\underline{E}$.

# APPENDIX B

## COMPUTER PROGRAMMING CONSIDERATIONS

The ROMMAP and SUBMAP creation programs for the K = 5 and
K = 7 codes discussed in Section 5.1 and 5.2 were initially pro-
grammed on an IBM 360/44 system in FORTRAN IV. The necessary
conversion from SNR ($E_s/N_o$) to BSC crossover probability p was
done as explained in Section 5.1.2 using the double precision mathe-
matical function DERFC(x) [30].

Due to 360 core limitations and operating system reliability, the
maximum syndrome length for which SUBMAP look-up tables could
be created using the 360 was approximately LSYN = 13 (requiring
about 2 hours of CPU time). Consequently the SUBMAP creation and
performance bounding programs were converted to the KL-10 Digi-
tal Equipment Corporation system using FORTRAN-10 [31]. Since
FORTRAN-10 provides a very limited selection of mathematical
functions, the SNR-to-p conversion was implemented using a called
subroutine MERFC from the IMSL library of subroutines [32]. A
comparison of both ERFC functions was made to verify that no signi-
ficant differences existed over the range of SNR investigated in this
work.

The simulation programs described in Section 5.5 were all run
on the KL-10 system which could practically support syndrome
lengths of LYSN = 17 (equivalent to roughly 4 hours of CPU time).

Since simulation results could be sensitive to the quality of the pseudo-random number (PRN) generator used, two different generators were implemented and compared. These were:

(1) GGUB, a standard subroutine from the IMSL library [32] and (2) REGOL, a PRN subroutine coded at USC.

It was found that for the SNR range investigated (0 to 5 db), no appreciable difference in simulation results was obtained. Figure B. 1 shows comparison results for REGOL and GGUB for a typical rate 1/2 simulation with $10^6$ calls for each of the two error inputs. It can be seen that the simulation results for both PRN's coincide except at the higher SNR level of 5 db $E_s/N_o$. Thus in practice, higher SNR levels (e. g., 4 and 5 db) were simulated using greater than $10^6$ calls per input always insuring 100 or more actual errors were created.

For reference purposes a brief description of the REGOL subroutine as implemented by G. G. Finn and J. M. Smith of USC follows.

FUNCTION REGOL

AUTHORS
    G.G. FINN, IMAGE PROCESS. INST.
    J.M. SMITH, DEPT. ELEC. ENG.
    UNIV. SO. CALIF.

PURPOSE
    GENERATES A STANDARD UNIFORMLY DISTRIBUTED
    RANDOM NUMBER (UNIFORM ON (0, 1)).

USAGE
    X = REGOL(IR)

DESCRIPTION OF PARAMETERS
    IR      - INPUT.  FOR THE FIRST ENTRY THIS VARIABLE
                MUST BE INITIALIZED TO ANY ODD INTEGER IN
                THE EXCLUSIVE RANGE (0, 34359738368).  UPON
                RETURN IT WILL CONTAIN A NEW ODD INTEGER
                TO BE USED ON THE NEXT ENTRY TO THE SUB-
                PROGRAM.  THIS VARIABLE MUST NEVER BE
                REDEFINED BY THE CALLING PROGRAM;
                OTHERWISE THE SEQUENCE OF RANDOM NUMBERS
                WILL BE DISTURBED.
    REGOL   - OUTPUT.  STANDARD UNIFORMLY DISTRIBUTED
                REAL RANDOM NUMBER.

REMARKS
    THIS SUBPROGRAM IS SPECIFIC TO THE DECSYSTEM-10
    AND DECSYSTEM-20 SERIES OF COMPUTERS.

METHOD
    THIS SUBPROGRAM REALIZES THE MULTIPLICATIVE
    CONGRUENTIAL METHOD (PERIOD 2**33) USING A MULTI-
    PLIER CHOSEN ACCORDING TO THE RECOMMENDATION OF
    REFERENCE 1.

REFERENCE
    [1] J.H. AHRENS, U. DIETER, AND A. GRUBE, COMPUTING,
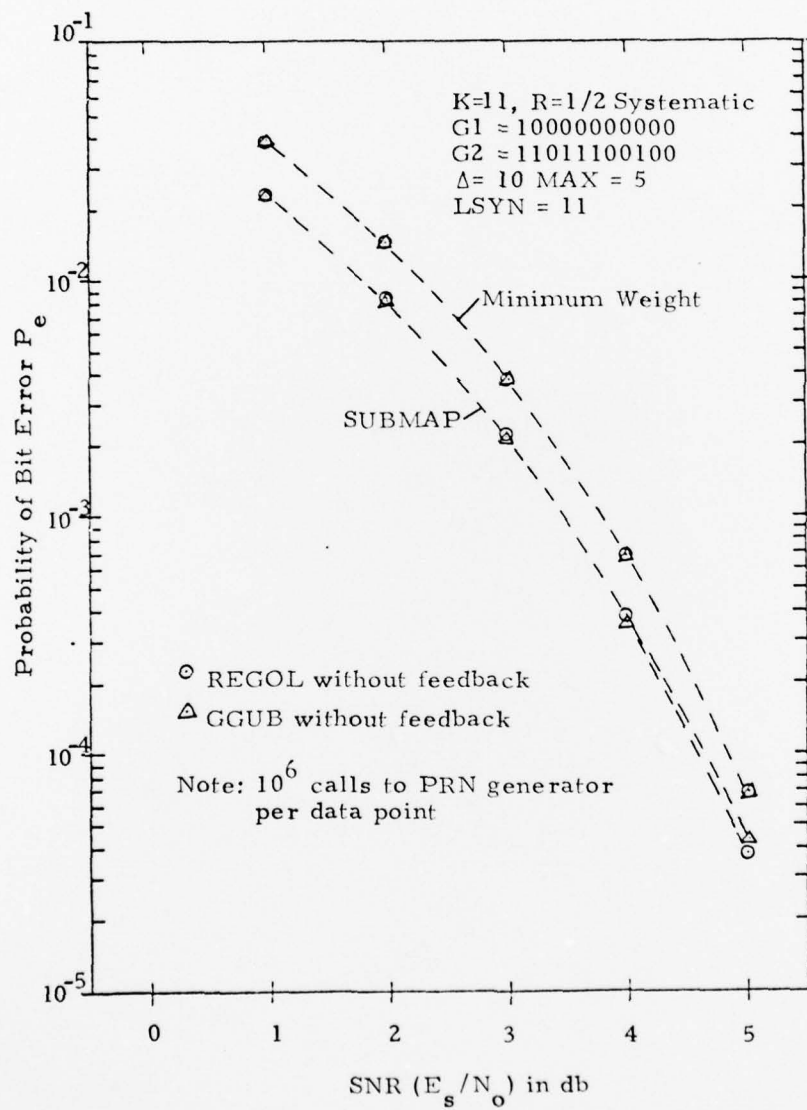        VOL. 6, PP. 121-138, 1970.

Figure B.1 Comparison of REGOL and GGUB PRN's in Simulation

## APPENDIX C

### OPTIMAL FINITE DELAY DECODING WHEN $\Delta \leqq \nu$

In the optimal decoding algorithm when $\Delta \leqq \nu$ as noted in Chapter 3 Section 3.3, there is a branch from Step 1 to Step 4 given below.

-------------------------------------------------------------

Step 4: (Final average and decision)

For $i = 0, 1$ calculate

$$\Lambda_i = \sum_{e_{t-\nu+1}} \cdots \sum_{e_t} q(e_{t-\nu} = i, e_{(t-\nu+1, t)})$$

If $\Lambda_0 > \Lambda_1$, $\hat{e}_{t-\nu} = 0$, otherwise $e_{t-\nu} = 1$.

Continue with Step 5.

-------------------------------------------------------------

For illustration, consider $t = \nu+1$. Step 4 yields

$$\Lambda_0 = \sum_{e_2} \cdots \sum_{e_{\nu+1}} q(e_1 = 0, e_{(2, \nu+1)}) = \left\{ \begin{array}{l} \text{sum of all paths in error} \\ \text{tree with } e_1 = 0, \text{ i.e., the} \\ \text{lower half tree.} \end{array} \right\}$$

$$\Lambda_1 = \sum_{e_2} \cdots \sum_{e_{\nu+1}} q(e_1 = 1, e_{(2, \nu+1)}) = \left\{ \begin{array}{l} \text{sum of all paths in error} \\ \text{tree with } e_1 = 1, \text{ i.e., the} \\ \text{upper half tree.} \end{array} \right\}$$

But there is a number for each $q(e_{(1, \nu+1)})$ path in the error tree. Thus Step 4 merely requires the adding together of the $q(e_{(1, \nu+1)})$'s at the end of the appropriate half trees to make the decision on $\hat{e}_{t-\nu}$ (in this case on $\hat{e}_1$).

For completeness, Step 5 following Step 4 is given below.

------------------------------------------------------------------

Step 5: (following Step 4 for case of $\Delta \leqq \nu$).

Increase t by 1.

Calculate

$$q(e_{(t-\nu, t)}) = P(e_t)P(Z_t/e_{(t-\nu, t)}) \sum_{e_{t-\nu-1}} q(e_{(t-\nu-1, t-1)})$$

Return to Step 1.

------------------------------------------------------------------

Step 5 would cause $t = \nu+1$ to become $t = \nu+2$ and

$$\underbrace{q(e_{(2, \nu+2)})}_{\substack{2^{\nu+1} \text{ values} \\ \text{in this set}}} = P(e_{\nu+2})P(Z_{\nu+2}/e_{(2, \nu+2)}) \sum_{e_1} q(e_{(1, \nu+1)})$$

(average over $e_1$)

$$= P(e_{\nu+2})P(Z_{\nu+2}/e_{(2, \nu+2)}) \begin{bmatrix} q(e_1 = 0, e_{(2, \nu+1)})P(e_1 = 0) + \\ \\ q(e_1 = 1, e_{(2, \nu+1)})P(e_1 = 1) \end{bmatrix}$$

Here one branch in the upper half tree has been added to the corresponding path in the lower half tree with each path weighted by $P(e_1 = 0)$ or $P(e_1 = 1)$. This results in a new tree as shown in Figure C.1.

------------------------------------------------------------------

Step 1: (following Step 5 when $\Delta \leqq \nu$)

Set $P(e_{t-\nu}, Z_{(1, t)}, e_{(t-\nu+1, t)}) = q(e_{(t-\nu, t)})$

Go to Step 4 to continue.

------------------------------------------------------------------

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Here, with $t = \nu + 2$ now, one would have:

$$\underbrace{P(e_2, Z_{(1, \nu+2)}, e_{(3, \nu+2)})}_{2^{\nu+1} \text{ values}} \Leftarrow \underbrace{q(e_{(2, \nu+2)})}_{2^{\nu+1} \text{ values}}$$

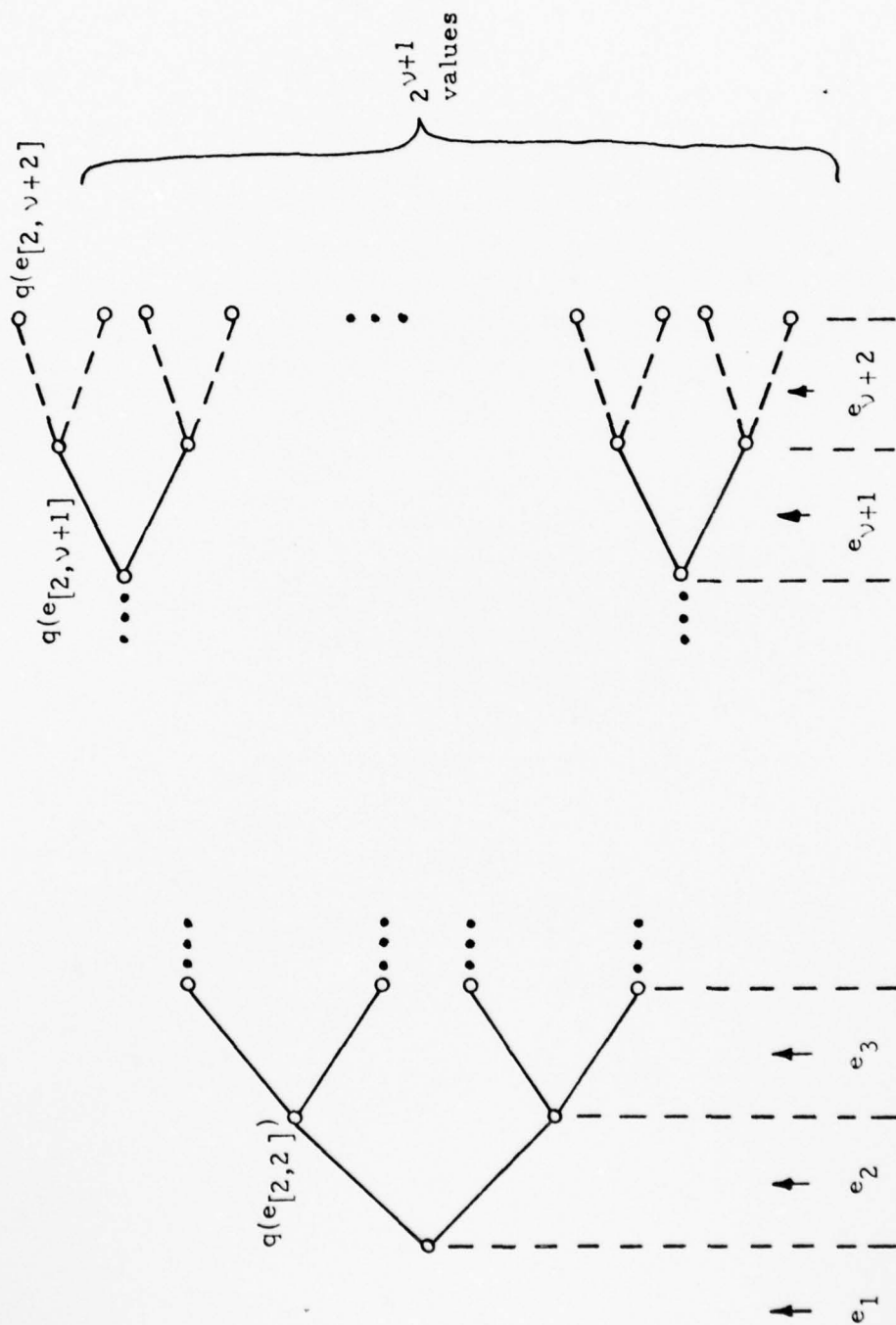and the algorithm would continue by going to Step 4 again.

Figure C.1 New Error Tree from Steps 5 in case $\Delta \leq \nu$

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